2-TRACK ALGEBRAS AND THE ADAMS SPECTRAL SEQUENCE

HANS-JOACHIM BAUES AND MARTIN FRANKLAND

Dedicated to Ronnie Brown on the occasion of his eightieth birthday.

Abstract. In previous work of the first author and Jibladze, the E_3-term of the Adams spectral sequence was described as a secondary derived functor, defined via secondary chain complexes in a groupoid-enriched category. This led to computations of the E_3-term using the algebra of secondary cohomology operations. In work with Blanc, an analogous description was provided for all higher terms E_r. In this paper, we introduce 2-track algebras and tertiary chain complexes, and we show that the E_4-term of the Adams spectral sequence is a tertiary Ext group in this sense. This extends the work with Jibladze, while specializing the work with Blanc in a way that should be more amenable to computations.

Contents

1.	Introduction	1
2.	Cubes and tracks in a space	3
3.	2-track groupoids	4
4.	2-tracks in a topologically enriched category	10
5.	2-track algebras	13
6.	Higher order chain complexes	14
7.	The Adams differential d_3	17
8.	Higher order resolutions	20
9.	Algebras of left 2-cubical balls	21
Appendix A.	Models for homotopy 2-types	23
References		29

1. Introduction

A major problem in algebraic topology consists of computing homotopy classes of maps between spaces or spectra, notably the stable homotopy groups of spheres $\pi_*^S(S^0)$. One of the most useful tools for such computations is the Adams spectral sequence [1] (and its unstable analogues [8]), based on ordinary mod p cohomology. Given finite spectra X and Y, Adams constructed a spectral sequence of the form:

$$E_2^{s,t} = \mathrm{Ext}_{\mathfrak{A}}^{s,t}\left(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)\right) \Rightarrow [\Sigma^{t-s}X, Y_p^\wedge]$$

Date: January 7, 2016.

2010 *Mathematics Subject Classification.* Primary: 55T15; Secondary: 18G50, 55S20.

Key words and phrases. Adams spectral sequence, tertiary cohomology operation, tertiary chain complex, tertiary Ext-group, Toda bracket, 2-track algebra, 2-track groupoid, bigroupoid, double groupoid, cubical set.

where $\mathfrak{A}$ is the mod p Steenrod algebra, consisting of primary stable mod p cohomology operations, and $Y_p^\wedge$ denotes the p-completion of Y. In particular, taking sphere spectra $X = Y = S^0$, one obtains a spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_{\mathfrak{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{t-s}^S(S^0)_p^\wedge$$

abutting to the p-completion of the stable homotopy groups of spheres. The differential d_r is determined by r^{th} order cohomology operations [14]. In particular, secondary cohomology operations determine the differential d_2 and thus the E_3-term. The algebra of secondary operations was studied in [2]. In [3], the first author and Jibladze developed secondary chain complexes and secondary derived functors, and showed that the Adams E_3-term is given by secondary Ext groups of the secondary cohomology of X and Y. They used this in [5], along with the algebra of secondary operations, to construct an algorithm that computes the differential d_2.

Primary operations in mod p cohomology are encoded by the homotopy category $\operatorname{Ho}(\mathcal{K})$ of the Eilenberg-MacLane mapping theory $\mathcal{K}$, consisting of finite products of Eilenberg-MacLane spectra of the form $\Sigma^{n_1} H\mathbb{F}_p \times \cdots \times \Sigma^{n_k} H\mathbb{F}_p$. More generally, the n^{th} Postnikov truncation $P_n\mathcal{K}$ of the Eilenberg-MacLane mapping theory encodes operations of order up to $n+1$, which in turn determine the Adams differential d_{n+1} and thus the E_{n+2}-term [4]. However, $P_n\mathcal{K}$ contains too much information for practical purposes. In [6], the first author and Blanc extracted from $P_n\mathcal{K}$ the information needed in order to compute the Adams differential d_{n+1}. The resulting algebraic-combinatorial structure is called an *algebra of left n-cubical balls*.

In this paper, we specialize the work of [6] to the case $n = 2$. Our goal is to provide an alternate structure which encodes an algebra of left 2-cubical balls, but which is more algebraic in nature and better suited for computations. The combinatorial difficulties in an algebra of left n-cubical balls arise from triangulations of the sphere $S^{n-1} = \partial D^n$. In the special case $n = 2$, triangulations of the circle S^1 are easily described, unlike in the case $n > 2$. Our approach also extends the work in [3] from secondary chain complexes to tertiary chain complexes.

Organization and main results. We define the notion of 2-track algebra (Definition 5.1) and show that each 2-track algebra naturally determines an algebra of left 2-cubical balls (Theorem 9.3). Building on [6], we show that higher order resolutions always exist in a 2-track algebra (Theorem 8.7). We show that a suitable 2-track algebra related to the Eilenberg-MacLane mapping theory recovers the Adams spectral sequence up to the E_4-term (Theorem 7.3). We show that the spectral sequence only depends on the weak equivalence class of the 2-track algebra (Theorem 7.5).

Remark 1.1. This last point is important in view of the strictification result for secondary cohomology operations: these can be encoded by a graded pair algebra B_* over $\mathbb{Z}/p^2$ [2, §5.5]. The secondary Ext groups of the E_3-term turn out to be the usual Ext groups over B_* [5, Theorem 3.1.1], a key fact for computations. We conjecture that a similar strictification result holds for tertiary operations, i.e., in the case $n = 2$.

Appendix A explains why 2-track groupoids are not models for homotopy 2-types, and how to extract the underlying 2-track groupoid from a bigroupoid or a double groupoid.

Acknowledgments. We thank the referee for their helpful comments. The second author thanks the Max-Planck-Institut für Mathematik Bonn for its generous hospitality, as well as David Blanc, Robert Bruner, Dan Christensen, and Dan Isaksen for useful conversations.

2. Cubes and tracks in a space

In this section, we fix some notation and terminology regarding cubes and groupoids.

Definition 2.1. Let X be a topological space.

An n-**cube** in X is a map $a\colon I^n \to X$, where $I = [0,1]$ is the unit interval. For example, a 0-cube in X is a point of X, and a 1-cube in X is a path in X.

An n-cube can be restricted to $(n-1)$-cubes along the $2n$ faces of I^n. For $1 \leq i \leq n$, denote:

$$d_i^0(a) = a \text{ restricted to } I \times I \times \ldots \times \overbrace{\{0\}}^{i} \times \ldots \times I$$
$$d_i^1(a) = a \text{ restricted to } I \times I \times \ldots \times \overbrace{\{1\}}^{i} \times \ldots \times I.$$

An n-**track** in X is a homotopy class, relative to the boundary ∂I^n, of an n-cube. If $a\colon I^n \to X$ is an n-cube in X, denote by $\{a\}$ the corresponding n-track in X, namely the homotopy class of a rel ∂I^n.

In particular, for $n = 1$, a 1-track $\{a\}$ is a path homotopy class, i.e., a morphism in the fundamental groupoid of X from $a(0)$ to $a(1)$. Let us fix our notation regarding groupoids. In this paper, we consider only *small* groupoids.

Notation 2.2. A **groupoid** is a (small) category in which every morphism is invertible. Denote the data of a groupoid by $G = \left(G_0, G_1, \delta_0, \delta_1, \mathrm{id}^{\square}, \square, (-)^{\boxminus}\right)$, where:

- $G_0 = \mathrm{Ob}(G)$ is the set of objects of G.
- $G_1 = \mathrm{Hom}(G)$ is the set of morphisms of G. The set of morphisms from x to y is denoted $G(x,y)$. We write $x \in G$ and $\deg(x) = 0$ for $x \in G_0$, and $\deg(x) = 1$ for $x \in G_1$.
- $\delta_0\colon G_1 \to G_0$ is the source map.
- $\delta_1\colon G_1 \to G_0$ is the target map.
- $\mathrm{id}^{\square}\colon G_0 \to G_1$ sends each object x to its corresponding identity morphism $\mathrm{id}_x^{\square}$.
- $\square\colon G_1 \times_{G_0} G_1 \to G_1$ is composition in G.
- $f^{\boxminus}\colon y \to x$ is the inverse of the morphism $f\colon x \to y$.

Groupoids form a category **Gpd**, where morphisms are functors between groupoids.

For any object $x \in G_0$, denote by $\mathrm{Aut}_G(x) = G(x,x)$ the automorphism group of x.

Denote by $\mathrm{Comp}(G) = \pi_0(G)$ the components of G, i.e., the set of isomorphism classes of objects $G_0/\sim$.

Denote the fundamental groupoid of a topological space X by $\Pi_{(1)}(X)$.

Definition 2.3. Let X be a pointed space, with basepoint $0 \in X$. The constant map $0\colon I^n \to X$ with value $0 \in X$ is called the **trivial** n-**cube**.

A **left** 1-**cube** or **left path** in X is a map $a\colon I \to X$ satisfying $a(1) = 0$, that is, $d_1^1(a) = 0$, the trivial 0-cube. In other words, a is a path in X from a point $a(0)$ to the basepoint 0. We denote $\delta a = a(0)$.

A **left** 2-**cube** in X is a map $\alpha\colon I^2 \to X$ satisfying $\alpha(1,t) = \alpha(t,1) = 0$ for all $t \in I$, that is, $d_1^1(\alpha) = d_2^1(\alpha) = 0$, the trivial 1-cube.

More generally, a **left** n-**cube** in X is a map $\alpha\colon I^n \to X$ satisfying $\alpha(t_1, \ldots, t_n) = 0$ whenever some coordinate satisfies $t_i = 1$. In other words, for all $1 \leq i \leq n$ we have $d_i^1(\alpha) = 0$, the trivial $(n-1)$-cube.

A **left n-track** in X is a homotopy class, relative to the boundary ∂I^n, of a left n-cube.

The equality $I^{m+n} = I^m \times I^n$ allows us to define an operation on cubes.

Definition 2.4. Let $\mu\colon X \times X' \to X''$ be a map, for example a composition map in a topologically enriched category $\mathcal{C}$. For $m, n \geq 0$, consider cubes

$$a\colon I^m \to X$$
$$b\colon I^n \to X'.$$

The **$\otimes$-composition** of a and b is the $(m+n)$-cube $a \otimes b$ defined as the composite

$$a \otimes b\colon I^{m+n} = I^m \times I^n \xrightarrow{a\times b} X \times X' \xrightarrow{\mu} X''. \tag{2.1}$$

For $m = n$, the **pointwise composition** of a and b is the n-cube defined as the composite

$$ab\colon I^n \xrightarrow{(a,b)} X \times X' \xrightarrow{\mu} X''. \tag{2.2}$$

The pointwise composition is the restriction of the $\otimes$-composition along the diagonal:

$$I^n \xrightarrow{\Delta} I^n \times I^n \xrightarrow{a\otimes b} X'', \qquad ab = (a\otimes b)\circ\Delta\colon I^n \to X''.$$

Remark 2.5. For $m = n = 0$, the 0-cube $x \otimes y = xy$ is the pointwise composition, which is the composition in the underlying category. For higher dimensions, there are still relations between the $\otimes$-composition and the pointwise composition. In suggestive formulas, pointwise composition of paths is given by $(ab)(t) = a(t)b(t)$ for all $t \in I$, whereas the $\otimes$-composition of paths is the 2-cube given by $(a \otimes b)(s,t) = a(s)b(t)$.

Assume moreover that μ satisfies

$$\mu(x, 0) = \mu(0, x') = 0$$

for the basepoints $0 \in X, 0 \in X', 0 \in X''$. For example, μ could be the composition map in a category $\mathcal{C}$ enriched in $(\mathbf{Top}_*, \wedge)$, the category of pointed topological spaces with the smash product as monoidal structure. If a and b are left cubes, then $a \otimes b$ and ab are also left cubes.

3. 2-track groupoids

We now focus on left 2-tracks in a pointed space X, and observe that they form a groupoid. Define the groupoid $\Pi_{(2)}(X)$ with object set:

$$\Pi_{(2)}(X)_0 = \text{ set of left 1-cubes in } X$$

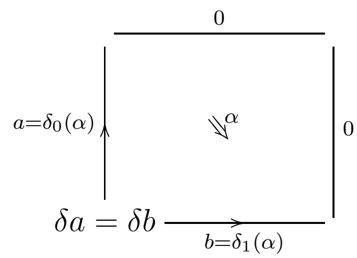
and morphism set:

$$\Pi_{(2)}(X)_1 = \text{ set of left 2-tracks in } X$$

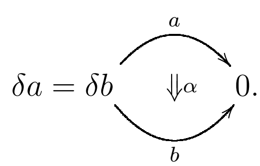
where the source δ_0 and target δ_1 of a left 2-track $\alpha\colon I \times I \to X$ are given by restrictions

$$\delta_0(\alpha) = d_1^0(\alpha)$$
$$\delta_1(\alpha) = d_2^0(\alpha)$$

and note in particular $\delta\delta_0(\alpha) = \delta\delta_1(\alpha) = \alpha(0,0)$. In other words, a morphism α from a to b looks like this:



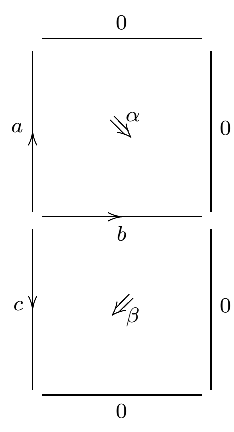
Remark 3.1. Up to reparametrization, a left 2-track $\alpha\colon a \Rightarrow b$ corresponds to a path homotopy from a to b, which can be visualized in a globular picture:



However, the $\otimes$-composition will play an important role in this paper, which is why we adopt a cubical approach, rather than globular or simplicial.

Composition $\beta \square \alpha$ of left 2-tracks is described by the following picture:

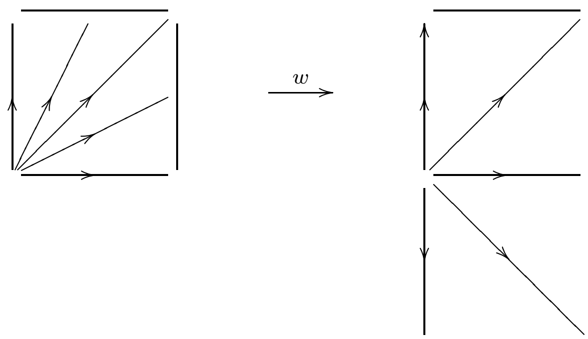
(3.1)



Remark 3.2. To make this definition precise, let $\alpha\colon a \Rightarrow b$ and $\beta\colon b \Rightarrow c$ be left 2-tracks in X, i.e., composable morphisms in $\Pi_{(2)}(X)$. Choose representative maps $\widetilde{\alpha}, \widetilde{\beta}\colon I^2 \to X$. Consider the map $f_{\alpha,\beta}\colon [0,1] \times [-1,1] \to X$ pictured in (3.1). That is, define

$$f(s,t) = \begin{cases} \widetilde{\alpha}(s,t) & \text{if } 0 \le t \le 1 \\ \widetilde{\beta}(-t,s) & \text{if } -1 \le t \le 0. \end{cases}$$

Now consider the reparametrization map $w\colon I^2 \to [0,1]\times[-1,1]$ illustrated in this picture:



Explicitly, the restriction $w|_{\partial I^2}$ to the boundary is the piecewise linear map satisfying

$$\begin{cases} w(0,0) = (0,0) \\ w(0,1) = (0,1) \\ w(\frac{1}{2},1) = (1,1) \\ w(1,1) = (1,0) \\ w(1,\frac{1}{2}) = (1,-1) \\ w(1,0) = (0,-1) \end{cases}$$

and $w(x)$ is defined for points $x \in I^2$ in the interior as follows. Write $x = p(0,0) + qy$ as a unique convex combination of $(0,0)$ and a point y on the boundary ∂I^2. Then define $w(x) = pw(0,0) + qw(y) = qw(y)$. Finally, the composition $\beta \Box \alpha\colon a \Rightarrow c$ is $\{f_{\alpha,\beta} \circ w\}$, the homotopy class of the composite

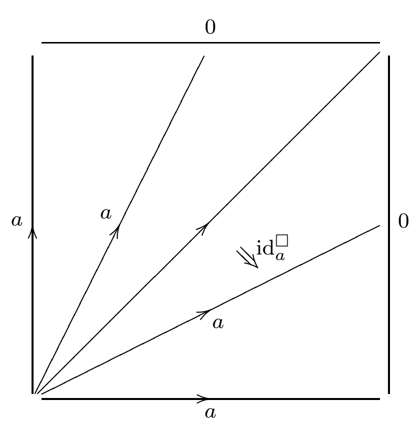
$$I^2 \xrightarrow{w} [0,1]\times[-1,1] \xrightarrow{f_{\alpha,\beta}} X$$

relative to the boundary ∂I^2.

In other notation, we have inclusions $d_2^0\colon I^1 \hookrightarrow I^2$ as the bottom edge $I \times \{0\}$ and $d_1^0\colon I^1 \hookrightarrow I^2$ as the left edge $\{0\}\times I$, our w is a map $w\colon I^2 \to I^2 \cup_{I^1} I^2$, and $\beta \Box \alpha$ is the homotopy class of the composite

$$I^2 \xrightarrow{w} I^2 \cup_{I^1} I^2 \xrightarrow{[\alpha\ \beta]} X.$$

Given a left path a in X, the identity of a in the groupoid $\Pi_{(2)}(X)$ is the left 2-track is pictured here:



More precisely, for points $x \in I^2$ in the interior, write $x = p(0,0) + qy$ as a unique convex combination of $(0,0)$ and a point y on the boundary ∂I^2. Then define $\mathrm{id}_a^{\square}(x) = a(q)$.

The inverse $\alpha^{\boxminus}\colon b \Rightarrow a$ of a left 2-track $\alpha\colon a \Rightarrow b$ is the homotopy class of the composite $\alpha \circ T$, where $T\colon I^2 \to I^2$ is the map swapping the two coordinates: $T(x,y) = (y,x)$.

Lemma 3.3. *Given a pointed topological space X, the structure described above makes $\Pi_{(2)}(X)$ into a groupoid, called the **groupoid of left 2-tracks** in X.*

Proof. Standard. □

Definition 3.4. A groupoid G is **abelian** if the group $\mathrm{Aut}_G(x)$ is abelian for every object $x \in G_0$. The groupoid G is **strictly abelian** if it is pointed (with basepoint $0 \in G_0$), and is equipped with a family of isomorphisms

$$\psi_x\colon\ \mathrm{Aut}_G(x) \xrightarrow{\cong} \mathrm{Aut}_G(0)$$

indexed by all objects $x \in G_0$, such that the diagram

$$\begin{array}{ccc} \mathrm{Aut}_G(y) & \xrightarrow{\varphi^f} & \mathrm{Aut}_G(x) \\ & {\scriptstyle \psi_y} \searrow & \downarrow {\scriptstyle \psi_x} \\ & & \mathrm{Aut}_G(0) \end{array} \tag{3.2}$$

commutes for every map $f\colon x \to y$ in G, where φ^f denotes the "change of basepoint" isomorphism

$$\begin{aligned} \varphi^f\colon\ \mathrm{Aut}_G(y) &\xrightarrow{\cong} \mathrm{Aut}_G(x) \\ \alpha &\mapsto \varphi^f(\alpha) = f^{\boxminus} \square\, \alpha \square\, f. \end{aligned}$$

Remark 3.5. A strictly abelian groupoid is automatically abelian. Indeed, the compatibility condition (3.2) applied to automorphisms $f\colon 0 \to 0$ implies that conjugation $\varphi^f\colon\ \mathrm{Aut}_G(0) \to \mathrm{Aut}_G(0)$ is the identity.

Definition 3.6. A groupoid G is **pointed** if it has a chosen basepoint, i.e., an object $0 \in G_0$. Here 0 is an abuse of notation: the basepoint is not assumed to be an initial object for G.

The **star** of a pointed groupoid G is the set of all morphisms to the basepoint 0, denoted by:

$$\mathrm{Star}(G) = \{ f \in G_1 \mid \delta_1(f) = 0 \}.$$

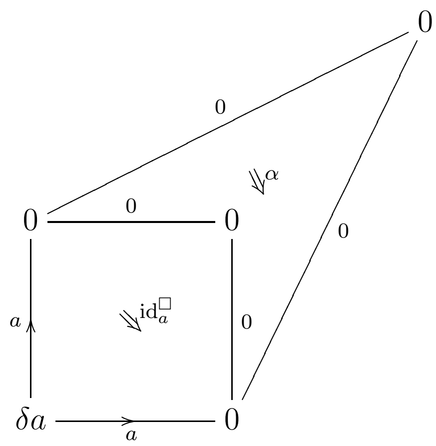
For a morphism $f\colon x \to 0$ in $\mathrm{Star}(G)$, we write $\delta f = \delta_0 f = x$.

If G has a basepoint $0 \in G_0$, then we take $\mathrm{id}_0^{\square} \in G_1$ as basepoint for the set of morphisms G_1 and for $\mathrm{Star}(G) \subseteq G_1$; we sometimes write $0 = \mathrm{id}_0^{\square}$. Moreover, we take the component of the basepoint 0 as basepoint for $\mathrm{Comp}(G)$, the set of components of G.

Proposition 3.7. $\Pi_{(2)}(X)$ *is a strictly abelian groupoid, and it satisfies* $\mathrm{Comp}\,\Pi_{(2)}(X) \simeq \mathrm{Star}\,\Pi_{(1)}(X)$.

Proof. Let $a \in \Pi_{(2)}(X)_0$ be a left path in X. To any automorphism $\alpha \colon 0 \Rightarrow 0$ in $\Pi_{(2)}(X)$, one can associate the well-defined left 2-track indicated by the picture

(3.3)



which is a morphism $a \Rightarrow a$. This assignment defines a map $\mathrm{Aut}_{\Pi_{(2)}(X)}(0) \to \mathrm{Aut}_{\Pi_{(2)}(X)}(a)$ and is readily seen to be a group isomorphism, whose inverse we denote ψ_a. One readily checks that the family ψ_a is compatible with change-of-basepoint isomorphisms.

The set $\mathrm{Comp}\,\Pi_{(2)}(X)$ is the set of left paths in X quotiented by the relation of being connected by a left 2-track. The set $\mathrm{Star}\,\Pi_{(1)}(X)$ is the set of left paths in X quotiented by the relation of path homotopy. But two left paths are path-homotopic if and only if they are connected by a left 2-track. □

The bijection $\mathrm{Comp}\,\Pi_{(2)}(X) \simeq \mathrm{Star}\,\Pi_{(1)}(X)$ is induced by taking the homotopy class of left 1-cubes. Consider the function $q \colon \Pi_{(2)}(X)_0 \to \Pi_{(1)}(X)_1$ which sends a left 1-cube to its left 1-track $q(a) = \{a\}$. Then the image of q is $\mathrm{Star}\,\Pi_{(1)}(X) \subseteq \Pi_{(1)}(X)_1$ and q is constant on the components of $\Pi_{(2)}(X)_0$. We now introduce a definition based on those features of $\Pi_{(2)}(X)$.

Definition 3.8. A **2-track groupoid** $G = (G_{(1)}, G_{(2)})$ consists of:

- Pointed groupoids $G_{(1)}$ and $G_{(2)}$, with $G_{(2)}$ strictly abelian.
- A pointed function $q \colon G_{(2)0} \twoheadrightarrow \mathrm{Star}\,G_{(1)}$ which is constant on the components of $G_{(2)}$, and such that the induced function $q \colon \mathrm{Comp}\,G_{(2)} \xrightarrow{\simeq} \mathrm{Star}\,G_{(1)}$ is bijective.

We assign degrees to the following elements:

$$\deg(x) = \begin{cases} 0 \text{ if } x \in G_{(1)0} \\ 1 \text{ if } x \in G_{(2)0} \\ 2 \text{ if } x \in G_{(2)1} \end{cases}$$

and we write $x \in G$ in each case.

A **morphism of 2-track groupoids** $F \colon G \to G'$ consists of a pair of pointed functors

$$F_{(1)} \colon G_{(1)} \to G'_{(1)}$$
$$F_{(2)} \colon G_{(2)} \to G'_{(2)}$$

satisfying the following two conditions.

(1) (*Structural isomorphisms*) For every object $a \in G_{(2)0}$, the diagram

$$\begin{array}{ccc} \mathrm{Aut}_{G_{(2)}}(a) & \xrightarrow{F_{(2)}} & \mathrm{Aut}_{G'_{(2)}}(F_{(2)}a) \\ \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \psi_{F_{(2)}a}} \\ \mathrm{Aut}_{G_{(2)}}(0) & \xrightarrow[F_{(2)}]{} & \mathrm{Aut}_{G'_{(2)}}(0') \end{array}$$

commutes.

(2) (*Quotient functions*) The diagram

$$\begin{array}{ccc} G_{(2)0} & \xrightarrow{F_{(2)}} & G'_{(2)0} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ \mathrm{Star}\, G_{(1)} & \xrightarrow[F_{(1)}]{} & \mathrm{Star}\, G'_{(1)} \end{array}$$

commutes.

Let $\mathbf{Gpd_{(1,2)}}$ denote the category of 2-track groupoids.

Remark 3.9. If $\alpha \colon a \Rightarrow b$ is a left 2-track in a space, then the left paths a and b have the same starting point $\delta a = \delta b$. This condition is encoded in the definition of 2-track groupoid. Indeed, if $\alpha \colon a \Rightarrow b$ is a morphism in $G_{(2)}$, then $a, b \in G_{(2)0}$ belong to the same component of $G_{(2)}$. Thus, we have $q(a) = q(b) \in \mathrm{Star}\, G_{(1)}$ and in particular $\delta q(a) = \delta q(b) \in G_{(1)0}$.

Definition 3.10. The **fundamental** 2**-track groupoid** of a pointed space X is

$$\Pi_{(1,2)}(X) := \left(\Pi_{(1)}(X), \Pi_{(2)}(X)\right).$$

This construction defines a functor $\Pi_{(1,2)} \colon \mathbf{Top}_* \to \mathbf{Gpd_{(1,2)}}$.

Remark 3.11. The grading on $\Pi_{(1,2)}(X)$ defined in 3.8 corresponds to the dimension of the cubes. For $x \in \Pi_{(1,2)}(X)$, we have $\deg(x) = 0$ if x is a point in X, $\deg(x) = 1$ if x is a left path in X, and $\deg(x) = 2$ if x is a left 2-track in X. This 2-graded set is the left 2-cubical set $\mathrm{Nul}_2(X)$ [6, Definition 1.9].

Definition 3.12. Given a 2-track groupoid G, its **homotopy groups** are

$$\begin{aligned} \pi_0 G &= \mathrm{Comp}\, G_{(1)} \\ \pi_1 G &= \mathrm{Aut}_{G_{(1)}}(0) \\ \pi_2 G &= \mathrm{Aut}_{G_{(2)}}(0). \end{aligned}$$

Note that $\pi_0 G$ is a priori only a pointed set, $\pi_1 G$ is a group, and $\pi_2 G$ is an abelian group.

A morphism $F \colon G \to G'$ of 2-track groupoids is a **weak equivalence** if it induces an isomorphism on homotopy groups.

Remark 3.13. Let X be a topological space with basepoint $x_0 \in X$. Then the homotopy groups of its fundamental 2-track groupoid $G = \Pi_{(1,2)}(X, x_0)$ are the homotopy groups of the space $\pi_i G = \pi_i(X, x_0)$ for $i = 0, 1, 2$.

The following two lemmas are straightforward.

Lemma 3.14. $\mathbf{Gpd_{(1,2)}}$ *has products, given by* $G \times G' = \left(G_{(1)} \times G'_{(1)}, G_{(2)} \times G'_{(2)}\right)$*, and where the structural isomorphisms*

$$\psi_{(x,x')} \colon \operatorname{Aut}_{G_{(2)} \times G'_{(2)}}((x,x')) \xrightarrow{\simeq} \operatorname{Aut}_{G_{(2)} \times G'_{(2)}}((0,0'))$$

are given by $\psi_x \times \psi_{x'}$*, and the quotient function*

$$\begin{array}{l} (G \times G')_{(2)0} = G_{(2)0} \times G'_{(2)0} \\ \Big\downarrow {\scriptstyle q \times q'} \\ \operatorname{Star}(G \times G')_{(1)} = \operatorname{Star} G_{(1)} \times \operatorname{Star} G'_{(1)} \end{array}$$

is the product of the quotient functions for G *and* G'*.*

Lemma 3.15. *The fundamental* 2*-track groupoid preserves products:*

$$\Pi_{(1,2)}(X \times Y) \cong \Pi_{(1,2)}(X) \times \Pi_{(1,2)}(Y).$$

4. 2-tracks in a topologically enriched category

Throughout this section, let $\mathcal{C}$ be a category enriched in $(\mathbf{Top}_*, \wedge)$. Explicitly:

- For any objects A and B of $\mathcal{C}$, there is a morphism space $\mathcal{C}(A,B)$ with basepoint denoted $0 \in \mathcal{C}(A,B)$.
- For any objects A, B, and C, there is a composition map
$$\mu \colon \mathcal{C}(B,C) \times \mathcal{C}(A,B) \to \mathcal{C}(A,C)$$
which is associative and unital.
- Composition satisfies
$$\mu(x,0) = \mu(0,y) = 0$$
for all x and y.

We write $x \in \mathcal{C}$ if $x \in \mathcal{C}(A,B)$ for some objects A and B. For $x, y \in \mathcal{C}$, we write $xy = \mu(x,y)$ when x and y are composable, i.e., when the target of y is the source of x. From now on, whenever an expression such as xy or $x \otimes y$ appears, it is understood that x and y must be composable.

By Definition 2.4, we have the $\otimes$-composition $x \otimes y$ for $x, y \in \Pi_{(1)}\mathcal{C}$ and $\deg(x) + \deg(y) \leq 1$. For $\deg(a) = \deg(b) = 1$, we have:

$$\begin{aligned} ab &= (a \otimes \delta_1 b) \,\square\, (\delta_0 a \otimes b) \\ &= (\delta_1 a \otimes b) \,\square\, (a \otimes \delta_0 b)\,. \end{aligned}$$

This equation holds in any category enriched in groupoids, where ab denotes the (pointwise) composition. Note that for paths $\widetilde{a}$ and $\widetilde{b}$ representing a and b, the boundary of the 2-cube $\widetilde{a} \otimes \widetilde{b}$ corresponds to the equation.

Conversely, the $\otimes$-composition in $\Pi_{(1)}\mathcal{C}$ is determined by the pointwise composition. For $\deg(x) = \deg(y) = 0$ and $\deg(a) = 1$, we have:

$$\begin{cases} x \otimes y = xy \\ x \otimes a = \mathrm{id}_x^{\square} a \\ a \otimes x = a\mathrm{id}_x^{\square}\,. \end{cases} \tag{4.1}$$

We now consider the 2-track groupoids $\Pi_{(1,2)}\mathcal{C}(A,B)$ of morphism spaces in $\mathcal{C}$, and we write $x \in \Pi_{(1,2)}\mathcal{C}$ if $x \in \Pi_{(1,2)}\mathcal{C}(A,B)$ for some objects A, B of $\mathcal{C}$. By Definition 2.4, composition in $\mathcal{C}$ induces the $\otimes$-composition:

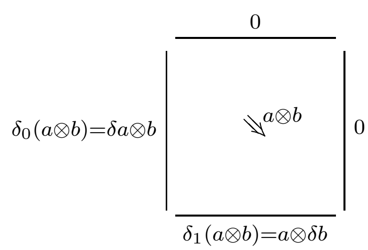
$$x \otimes y \in \Pi_{(1,2)}\mathcal{C}$$

if x and y satisfy $\deg(x) + \deg(y) \leq 2$. For $\deg(x) = \deg(y) = 1$, x and y are left paths, hence $x \otimes y$ is well-defined. The $\otimes$-composition satisfies:

$$\deg(x \otimes y) = \deg(x) + \deg(y).$$

The $\otimes$-composition is associative, since composition in $\mathcal{C}$ is associative. The identity elements $1_A \in \mathcal{C}(A,A)$ for $\mathcal{C}$ provide identity elements $1 = 1_A \in \Pi_{(1,2)}\mathcal{C}(A,A)$, with $\deg(1_A) = 0$, and $x \otimes 1 = x = 1 \otimes x$.

Let us describe the $\otimes$-composition of left paths more explicitly. Given left paths a and b, then $a \otimes b$ is a 2-track from $\delta_0(a \otimes b) = (\delta a) \otimes b$ to $\delta_1(a \otimes b) = a \otimes (\delta b)$, as illustrated here:



Definition 4.1. The 2**-track algebra associated to** $\mathcal{C}$, denoted $\left(\Pi_{(1)}\mathcal{C}, \Pi_{(1,2)}\mathcal{C}, \square, \otimes\right)$, consists of the following data.

- $\Pi_{(1)}\mathcal{C}$ is the category enriched in pointed groupoids given by the fundamental groupoids $\left(\Pi_{(1)}\mathcal{C}(A,B), \square\right)$ of morphism spaces in $\mathcal{C}$, along with the $\otimes$-composition, which determines (and is determined by) the composition in $\Pi_{(1)}\mathcal{C}$.
- $\Pi_{(1,2)}\mathcal{C}$ is given by the collection of fundamental 2-track groupoids $\left(\Pi_{(1,2)}\mathcal{C}(A,B), \square\right)$ together with the $\otimes$-composition $x \otimes y$ for $x, y \in \Pi_{(1,2)}\mathcal{C}$ satisfying $\deg(x) + \deg(y) \leq 2$.

Proposition 4.2. *Let* $x, \alpha, \beta \in \Pi_{(1,2)}\mathcal{C}$ *with* $\deg(x) = 0$ *and* $\deg(\alpha) = \deg(\beta) = 2$. *Then the following equations hold:*

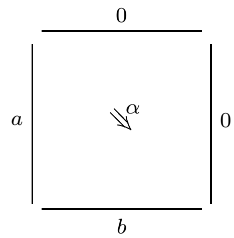
$$\begin{cases} x \otimes (\beta \square \alpha) = (x \otimes \beta) \square (x \otimes \alpha) \\ (\beta \square \alpha) \otimes x = (\beta \otimes x) \square (\alpha \otimes x). \end{cases}$$

Proof. This follows from functoriality of $\Pi_{(2)}$ applied to the composition maps $\mu(x,-)\colon \mathcal{C}(A,B) \to \mathcal{C}(A,C)$ and $\mu(-,x)\colon \mathcal{C}(B,C) \to \mathcal{C}(A,C)$. ∎

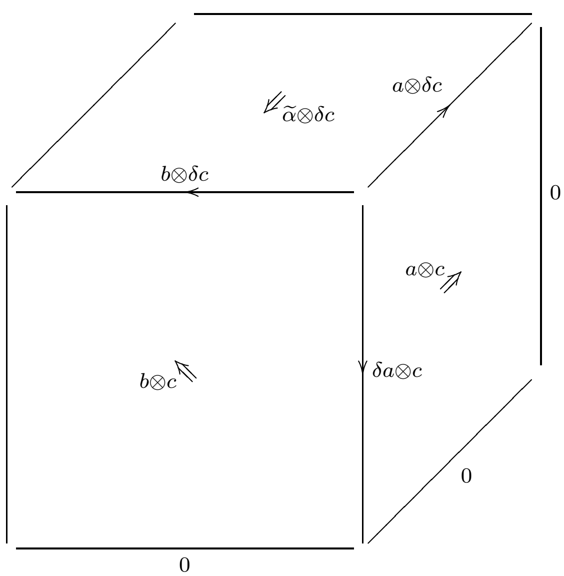
Proposition 4.3. *Let* $c, \alpha \in \Pi_{(1,2)}\mathcal{C}$ *with* $\deg(c) = 1$ *and* $\deg(\alpha) = 2$. *Then the following equations hold:*

$$\begin{cases} \delta_1\alpha \otimes c = (\alpha \otimes \delta c) \square (\delta_0\alpha \otimes c) \\ c \otimes \delta_0\alpha = (c \otimes \delta_1\alpha) \square (\delta c \otimes \alpha). \end{cases}$$

Proof. Write $a = \delta_0\alpha$ and $b = \delta_1\alpha$, i.e., α is a left 2-track from a to b:



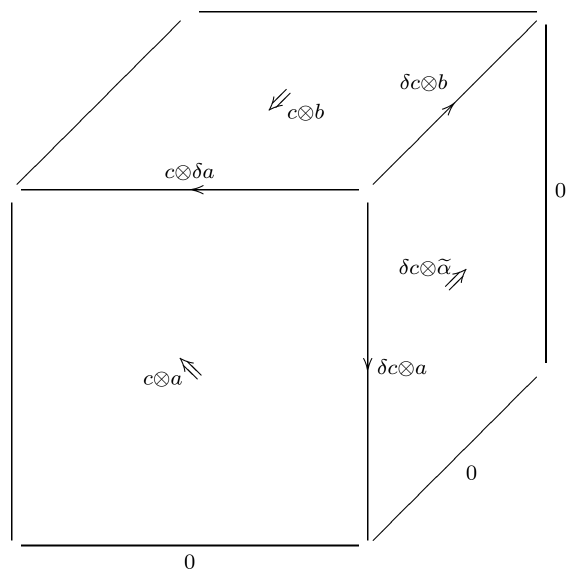
and note in particular $\delta a = \delta b$. Let $\widetilde{\alpha}$ be a left 2-cube that represents α and consider the left 3-cube $\widetilde{\alpha} \otimes c$:



Its boundary exhibits the equality of 2-tracks:

$$\begin{aligned} &\text{top face} \,\square\, \text{right face} \;=\; \text{front face} \\ &(\alpha \otimes \delta c) \,\square\, (a \otimes c) = b \otimes c \\ &(\alpha \otimes \delta c) \,\square\, (\delta_0\alpha \otimes c) = \delta_1\alpha \otimes c. \end{aligned}$$

Likewise, for second equation, consider the left 3-cube $c \otimes \widetilde{\alpha}$:



Its boundary exhibits the equality of 2-tracks:

$$\begin{aligned} &\text{top face} \,\square\, \text{right face} \;=\; \text{front face} \\ &(c \otimes b) \,\square\, (\delta c \otimes \alpha) = c \otimes a \\ &(c \otimes \delta_1 \alpha) \,\square\, (\delta c \otimes \alpha) = c \otimes \delta_0 \alpha. \end{aligned}$$

□

5. 2-track algebras

We now collect the structure found in $\left(\Pi_{(1)}\mathcal{C}, \Pi_{(1,2)}\mathcal{C}, \square, \otimes\right)$ into the following definition.

Definition 5.1. A **2-track algebra** $\mathcal{A} = \left(\mathcal{A}_{(1)}, \mathcal{A}_{(1,2)}, \square, \otimes\right)$ consists of the following data.

(1) A category $\mathcal{A}_{(1)}$ enriched in pointed groupoids, with the $\otimes$-composition determined by Equation (4.1).
(2) A collection $\mathcal{A}_{(1,2)}$ of 2-track groupoids $\left(\mathcal{A}_{(1,2)}(A, B), \square\right)$ for all objects A, B of $\mathcal{A}_{(1)}$, such that the first groupoid in $\mathcal{A}_{(1,2)}(A, B)$ is equal to the pointed groupoid $\mathcal{A}_{(1)}(A, B)$.
(3) For $x, y \in \mathcal{A}_{(1,2)}$, the $\otimes$-composition $x \otimes y \in \mathcal{A}_{(1,2)}$ is defined. For $\deg(x) = 0$ and $\deg(y) = 1$, the following equations hold in $\mathcal{A}_{(1)}$:

$$\begin{cases} q(x \otimes y) = x \otimes q(y) \\ q(y \otimes x) = q(y) \otimes x. \end{cases}$$

The following equations are required to hold.

(1) (*Associativity*) $\otimes$ is associative: $(x \otimes y) \otimes z = x \otimes (y \otimes z)$.
(2) (*Units*) The units $1 \in \mathcal{A}_{(1)}$, with $\deg(1_A) = 0$, serve as units for $\otimes$, i.e., satisfy $x \otimes 1 = x = 1 \otimes x$ for all $x \in \mathcal{A}_{(1,2)}$.
(3) (*Pointedness*) $\otimes$ satisfies $x \otimes 0 = 0$ and $0 \otimes y = 0$.
(4) For $x, y, \alpha, \beta \in \mathcal{A}_{(1,2)}$ with $\deg(x) = \deg(y) = 0$ and $\deg(\alpha) = \deg(\beta) = 2$, we have:

$$\begin{cases} \delta_i(x \otimes \alpha \otimes y) = x \otimes (\delta_i \alpha) \otimes y \;\; \text{for } i = 0, 1 \\ x \otimes (\beta \square \alpha) \otimes y = (x \otimes \beta \otimes y) \square (x \otimes \alpha \otimes y) \end{cases}$$

(5) For $a, b \in \mathcal{A}_{(1,2)}$ with $\deg(a) = \deg(b) = 1$, we have:

$$\begin{cases} \delta_0(a \otimes b) = \delta a \otimes b \\ \delta_1(a \otimes b) = a \otimes \delta b. \end{cases}$$

(6) For $c, \alpha \in \mathcal{A}_{(1,2)}$ with $\deg(c) = 1$ and $\deg(\alpha) = 2$, we have:

$$\begin{cases} \delta_1\alpha \otimes c = (\alpha \otimes \delta c) \,\square\, (\delta_0\alpha \otimes c) \\ c \otimes \delta_0\alpha = (c \otimes \delta_1\alpha) \,\square\, (\delta c \otimes \alpha)\,. \end{cases}$$

Definition 5.2. A **morphism of 2-track algebras** $F\colon \mathcal{A} \to \mathcal{B}$ consists of the following.

(1) A functor $F_{(1)}\colon \mathcal{A}_{(1)} \to \mathcal{B}_{(1)}$ enriched in pointed groupoids.
(2) A collection $F_{(1,2)}$ of morphisms of 2-track groupoids

$$F_{(1,2)}(A, B)\colon \mathcal{A}_{(1,2)}(A, B) \to \mathcal{B}_{(1,2)}(FA, FB)$$

for all objects A, B of $\mathcal{A}$, such that $F_{(1,2)}(A, B)$ restricted to the first groupoid in $\mathcal{A}_{(1,2)}(A, B)$ is the functor $F_{(1)}(A, B)\colon \mathcal{A}_{(1)}(A, B) \to \mathcal{B}_{(1)}(FA, FB)$.
(3) (*Compatibility with* $\otimes$) F commutes with $\otimes$:

$$F(x \otimes y) = Fx \otimes Fy.$$

Denote by $\mathbf{Alg}_{(\mathbf{1,2})}$ the category of 2-track algebras.

Definition 5.3. Let $\mathcal{A}$ be a 2-track algebra. The underlying **homotopy category** of $\mathcal{A}$ is the homotopy category of the underlying track category $\mathcal{A}_{(1)}$, denoted

$$\pi_0\mathcal{A} := \pi_0\mathcal{A}_{(1)} = \operatorname{Comp}\mathcal{A}_{(1)}.$$

We say that $\mathcal{A}$ is **based** on the category $\pi_0\mathcal{A}$.

Definition 5.4. A morphism of 2-track algebras $F\colon \mathcal{A} \to \mathcal{B}$ is a **weak equivalence** (or *Dwyer-Kan equivalence*) if the following conditions hold:

(1) For all objects A and B of $\mathcal{A}$, the morphism

$$F_{(1,2)}\colon \mathcal{A}_{(1,2)}(A, B) \to \mathcal{B}_{(1,2)}(FA, FB)$$

is a weak equivalence of 2-track groupoids (Definition 3.12).
(2) The induced functor $\pi_0 F\colon \pi_0\mathcal{A} \to \pi_0\mathcal{B}$ is an equivalence of categories.

6. Higher order chain complexes

In this section, we construct tertiary chain complexes, extending the work of [3] on secondary chain complexes. We will follow the treatment therein.

Definition 6.1. A **chain complex** (A, d) in a pointed category $\mathbf{A}$ is a sequence of objects and morphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_n} A_n \xrightarrow{d_{n-1}} A_{n-1} \longrightarrow \cdots$$

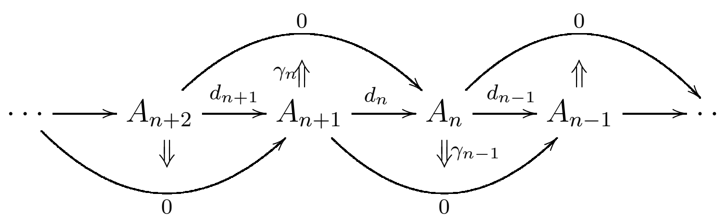
in $\mathbf{A}$ satisfying $d_{n-1}d_n = 0$ for all $n \in \mathbb{Z}$. The map d is called the *differential*.

A **chain map** $f\colon (A,d) \to (A',d')$ between chain complexes is a sequence of morphisms $f_n\colon A_n \to A'_n$ commuting with the differentials:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{d_n} & A_n & \xrightarrow{d_{n-1}} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & A'_{n+1} & \xrightarrow{d'_n} & A'_n & \xrightarrow{d'_{n-1}} & A'_{n-1} & \longrightarrow & \cdots
\end{array}$$

i.e., satisfying $f_n d_n = d'_n f_{n+1}$ for all $n \in \mathbb{Z}$.

Definition 6.2. [3, Definition 2.6] Let $\mathbf{B}$ be a category enriched in pointed groupoids. A **secondary pre-chain complex** (A,d,γ) in $\mathbf{B}$ is a diagram of the form:



More precisely, the data consists of a sequence of objects A_n and maps $d_n\colon A_{n+1} \to A_n$, together with left tracks $\gamma_n\colon d_n d_{n+1} \Rightarrow 0$ for all $n \in \mathbb{Z}$.

(A,d,γ) is a **secondary chain complex** if moreover for each $n \in \mathbb{Z}$, the tracks

$$d_{n-1}d_n d_{n+1} \xRightarrow{d_{n-1}\otimes\gamma_n} d_{n-1}0 \xRightarrow{\mathrm{id}_0^\square} 0$$

and

$$d_{n-1}d_n d_{n+1} \xRightarrow{\gamma_{n-1}\otimes d_{n+1}} 0 d_{n+1} \xRightarrow{\mathrm{id}_0^\square} 0$$

coincide. In other words, the track

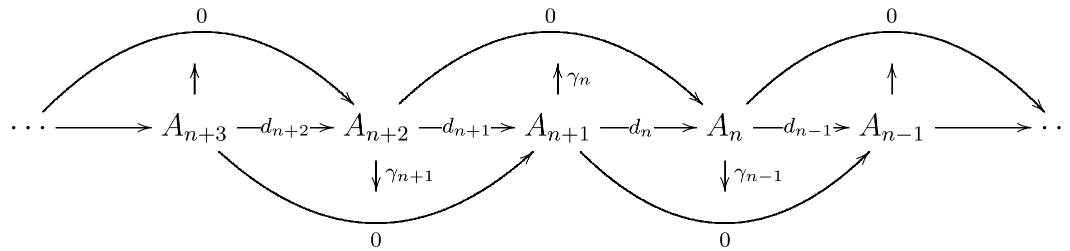
$$\mathcal{O}(\gamma_{n-1},\gamma_n) := (\gamma_{n-1}\otimes d_{n+1}) \,\square\, (d_{n-1}\otimes\gamma_n)^{\boxminus} : 0 \Rightarrow 0$$

in the groupoid $\mathbf{B}(A_{n+2}, A_{n-1})$ is the identity track of 0.

We say that the secondary pre-chain complex (A,d,γ) is **based** on the chain complex $(A,\{d\})$ in the homotopy category $\pi_0\mathbf{B}$.

Remark 6.3. One can show that the notion of secondary (pre-)chain complex in $\mathbf{B}$ coincides with the notion of 1^{st} *order (pre-)chain complex* in $\mathrm{Nul}_1\,\mathbf{B}$ described in [6, §4, c.f. Example 12.3].

Definition 6.4. A **tertiary pre-chain complex** (A,d,δ,ξ) in a 2-track algebra $\mathcal{A}$ is a sequence of objects A_n and maps $d_n\colon A_{n+1} \to A_n$ in the category $\mathcal{A}_{(1)0}$, together with left paths $\gamma_n \colon d_n d_{n+1} \to 0$ in $\mathcal{A}_{(1,2)}$, as illustrated in the diagram



along with left 2-tracks $\xi_n \colon \gamma_n \otimes d_{n+2} \Rightarrow d_n \otimes \gamma_{n+1}$ in $\mathcal{A}_{(1,2)}$, for all $n \in \mathbb{Z}$.

(A, d, γ, ξ) is a **tertiary chain complex** if moreover for each $n \in \mathbb{Z}$, the left 2-track:

$$d_{n-1} \otimes \gamma_n \otimes d_{n+2} \xRightarrow{d_{n-1} \otimes \xi_n} d_{n-1} d_n \otimes \gamma_{n+1} \xRightarrow{\gamma_{n-1} \otimes \gamma_{n+1}} \gamma_{n-1} \otimes d_{n+1} d_{n+2} \xRightarrow{\xi_{n-1} \otimes d_{n+2}} d_{n-1} \otimes \gamma_n \otimes d_{n+2}$$

is the identity of $d_{n-1} \otimes \gamma_n \otimes d_{n+2}$ in the groupoid $\mathcal{A}_{(2)}(A_{n+3}, A_{n-1})$. In other words, the element:

$$\begin{aligned} \mathcal{O}(\xi_{n-1}, \xi_n) := \psi_{d_{n-1} \otimes \gamma_n \otimes d_{n+2}} \left((\xi_{n-1} \otimes d_{n+2}) \,\square\, (\gamma_{n-1} \otimes \gamma_{n+1}) \,\square\, (d_{n-1} \otimes \xi_n) \right) \\ \in \pi_2 \mathcal{A}_{(1,2)}(A_{n+3}, A_{n-1}) \end{aligned}$$

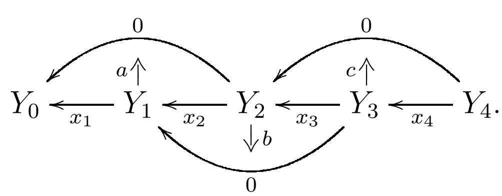
is trivial. Here, ψ is the structural isomorphism in the 2-track groupoid $\mathcal{A}_{(1,2)}(A_{n+3}, A_{n-1})$, as in Definitions 3.4 and 3.8.

We say that the tertiary pre-chain complex (A, d, γ, ξ) is **based** on the chain complex $(A, \{d\})$ in the homotopy category $\pi_0 \mathcal{A}$.

Toda brackets of length 3 and 4. Let $\mathcal{C}$ be a category enriched in $(\mathbf{Top}_*, \wedge)$. Let $\pi_0 \mathcal{C}$ be the category of path components of $\mathcal{C}$ (applied to each mapping space) and let

$$Y_0 \xleftarrow{y_1} Y_1 \xleftarrow{y_2} Y_2 \xleftarrow{y_3} Y_3 \xleftarrow{y_4} Y_4$$

be a diagram in $\pi_0 \mathcal{C}$ satisfying $y_1 y_2 = 0$, $y_2 y_3 = 0$, and $y_3 y_4 = 0$. Choose maps x_i in $\mathcal{C}$ representing y_i. Then there exist left 1-cubes a, b, c as in the diagram



Definition 6.5. The **Toda bracket** of length 3, denoted $\langle y_1, y_2, y_3 \rangle \subseteq \pi_1 \mathcal{C}(Y_3, Y_0)$, is the set of all elements in $\mathrm{Aut}(0) = \pi_1 \mathcal{C}(Y_3, Y_0)$ of the form

$$\mathcal{O}(a, b) := (a \otimes x_3) \square\, (x_1 \otimes b)^{\boxminus}$$

as above.

Assume now that we can choose left 2-tracks $\alpha \colon a \otimes x_3 \Rightarrow x_1 \otimes b$ and $\beta \colon b \otimes x_4 \Rightarrow x_2 \otimes c$ in $\Pi_{(1,2)} \mathcal{C}$. Then the composite of left 2-tracks

$$(\alpha \otimes x_4) \square\, (a \otimes c) \square\, (x_1 \otimes \beta)$$

is an element of $\mathrm{Aut}(x_1 \otimes b \otimes x_4)$, to which we apply the structural isomorphism

$$\psi_{x_1 \otimes b \otimes x_4} \colon \mathrm{Aut}(x_1 \otimes b \otimes x_4) \xrightarrow{\cong} \pi_2 \mathcal{C}(Y_4, Y_0).$$

The set of all such elements is the **Toda bracket** of length 4, denoted $\langle y_1, y_2, y_3, y_4 \rangle \subseteq \pi_2 \mathcal{C}(Y_4, Y_0)$.

Note that the existence of α, resp. β, implies that the bracket $\langle y_1, y_2, y_3 \rangle$, resp. $\langle y_2, y_3, y_4 \rangle$ contains the zero element.

Remark 6.6. For a secondary pre-chain complex (A, d, γ), we have

$$\mathcal{O}(\gamma_{n-1}, \gamma_n) \in \langle d_{n-1}, d_n, d_{n+1} \rangle$$

for every $n \in \mathbb{Z}$. Likewise, for a tertiary pre-chain complex (A, d, γ, ξ), we have

$$\mathcal{O}(\xi_{n-1}, \xi_n) \in \langle d_{n-1}, d_n, d_{n+1}, d_{n+2} \rangle$$

for every $n \in \mathbb{Z}$. However, the vanishing of these Toda brackets does not guarantee the existence of a tertiary chain complex based on the chain complex $(A, \{d\})$. In a secondary chain complex (A, d, γ), these Toda brackets vanish in a *compatible* way, that is, the equations $\mathcal{O}(\gamma_{n-1}, \gamma_n) = 0$ and $\mathcal{O}(\gamma_n, \gamma_{n+1}) = 0$ involve the same left track $\gamma_n \colon d_n d_{n+1} \Rightarrow 0$.

7. The Adams differential d_3

Let **Spec** denote the topologically enriched category of spectra and mapping spaces between them. More precisely, start from a simplicial (or topological) model category of spectra, like that of Bousfield–Friedlander [9, §2], or symmetric spectra or orthogonal spectra [13], and take **Spec** to be the full subcategory of fibrant-cofibrant objects; c.f. [6, Example 7.3].

Let $H := H\mathbb{F}_p$ be the Eilenberg-MacLane spectrum for the prime p and let $\mathfrak{A} = H^*H$ denote the mod p Steenrod algebra. Consider the collection **EM** of all mod p generalized Eilenberg-MacLane spectra that are bounded below and of finite type, i.e., degreewise finite products $A = \prod_i \Sigma^{n_i} H$ with $n_i \in \mathbb{Z}$ and $n_i \geq N$ for some integer N for all i. Since the product is degreewise finite, the natural map $\bigvee_i \Sigma^{n_i} H \to \prod_i \Sigma^{n_i} H$ is an equivalence, so that the mod p cohomology H^*A is a free $\mathfrak{A}$-module. Moreover, the cohomology functor restricted to the full subcategory of **Spec** with objects **EM** yields an equivalence of categories in the diagram:

$$\begin{array}{ccc} \pi_0 \mathbf{Spec}^{\mathrm{op}} & \xrightarrow{H^*} & \mathbf{Mod}_{\mathfrak{A}} \\ \uparrow & & \uparrow \\ \pi_0 \mathbf{EM}^{\mathrm{op}} & \xrightarrow[\cong]{H^*} & \mathbf{Mod}_{\mathfrak{A}}^{\mathrm{fin}} \end{array}$$

where $\mathbf{Mod}_{\mathfrak{A}}^{\mathrm{fin}}$ denotes the full subcategory consisting of free $\mathfrak{A}$-modules which are bounded below and of finite type.

Given spectra Y and X, consider the Adams spectral sequence:

$$E_2^{s,t} = \mathrm{Ext}_{\mathfrak{A}}^{s,t}(H^*X, H^*Y) \Rightarrow \left[\Sigma^{t-s} Y, X_p^\wedge\right].$$

Assume that Y is a finite spectrum and X is a connective spectrum of finite type, i.e., X is equivalent to a CW-spectrum with finitely many cells in each dimension and no cells below a certain dimension. Then the mod p cohomology H^*X is an $\mathfrak{A}$-module which is bounded below and degreewise finitely generated (as an $\mathfrak{A}$-module, or equivalently, as an $\mathbb{F}_p$-vector space). Choose a free resolution of H^*X as an $\mathfrak{A}$-module:

$$\cdots \longrightarrow F_2 \xrightarrow{e_1} F_1 \xrightarrow{e_0} F_0 \xrightarrow{\lambda} H^*X$$

where each F_i is a free $\mathfrak{A}$-module of finite type and bounded below. This diagram can be realized as the cohomology of a diagram in the stable homotopy category $\pi_0\mathbf{Spec}$:

$$\cdots \longleftarrow A_2 \xleftarrow{d_1} A_1 \xleftarrow{d_0} A_0 \xleftarrow{\epsilon} A_{-1} = X$$

with each A_i in $\mathbf{EM}$ (for $i \geq 0$) and satisfying $H^*A_i \cong F_i$. We consider this diagram as a diagram in the opposite category $\pi_0\mathbf{Spec}^{\mathrm{op}}$ of the form:

$$\cdots \longrightarrow A_2 \xrightarrow{d_1} A_1 \xrightarrow{d_0} A_0 \xrightarrow{\epsilon} A_{-1} = X$$

Since $A_\bullet \to X$ is an $\mathbf{EM}$-resolution of X in $\pi_0\mathbf{Spec}^{\mathrm{op}}$, there exists a tertiary chain complex (A, d, γ, ξ) in $\Pi_{(1,2)}\mathbf{Spec}^{\mathrm{op}}$ based on the resolution $A_\bullet \to X$, by Theorem 8.7.

Notation 7.1. Given spectra X and Y, let $\mathbf{EM}\{X, Y\}$ denote the topologically enriched subcategory of $\mathbf{Spec}$ consisting of all spectra in $\mathbf{EM}$ and mapping spaces between them, along with the objects X and Y, with the mapping spaces $\mathbf{Spec}(X, A)$ and $\mathbf{Spec}(Y, A)$ for all A in $\mathbf{EM}$; c.f. [3, Remark 4.3] [6, Remark 7.5]. We consider the 2-track algebra $\Pi_{(1,2)}\mathbf{EM}\{X, Y\}^{\mathrm{op}}$, or any 2-track algebra $\mathcal{A}$ weakly equivalent to it. In the following construction, everything will take place within $\Pi_{(1,2)}\mathbf{EM}\{X, Y\}^{\mathrm{op}}$, but we will write $\Pi_{(1,2)}\mathbf{Spec}^{\mathrm{op}}$ for notational convenience.

Start with a class in the E_2-term:

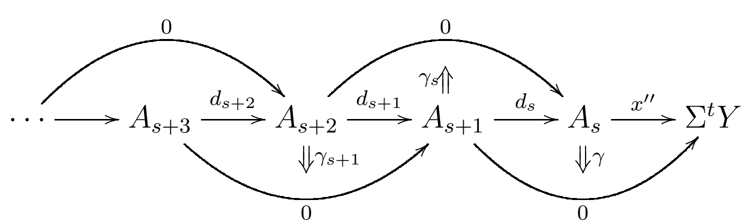
$$x \in E_2^{s,t} = \mathrm{Ext}_{\mathfrak{A}}^{s,t}(H^*X, H^*Y) = \mathrm{Ext}_{\mathfrak{A}}^{s,0}(H^*X, \Sigma^t H^*Y)$$

represented by a cocycle $x' \colon F_s \to \Sigma^t H^*Y$, i.e., a map of $\mathfrak{A}$-modules satisfying $x'd_s = 0$. Realize x' as the cohomology of a map $x'' \colon A_s \to \Sigma^t Y$ in $\mathbf{Spec}^{\mathrm{op}}$. The equation $x'd_s = 0$ means that $x''d_s$ is null-homotopic; let $\gamma \colon x''d_s \to 0$ be a null-homotopy. Consider the diagram in $\mathbf{Spec}^{\mathrm{op}}$:

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & A_{s+2} & \xrightarrow{d_{s+1}} & A_{s+1} & \xrightarrow{d_s} & A_s & \xrightarrow{d_{s-1}} & A_{s-1} & \longrightarrow \cdots \longrightarrow & A_0 & \xrightarrow{\epsilon} & X \\ & & & & & \downarrow{\scriptstyle x''} & & & & & & \\ & & & & & \Sigma^t Y & & & & & & \end{array}$$

Now consider the underlying secondary pre-chain complex in $\Pi_{(1)}\mathbf{Spec}^{\mathrm{op}}$:

(7.1)



in which the obstructions $\mathcal{O}(\gamma_i, \gamma_{i+1})$ are trivial, for $i \geq s$.

Theorem 7.2. *The obstruction $\mathcal{O}(\gamma, \gamma_s) \in \pi_1 \mathbf{Spec}^{\mathrm{op}}(A_{s+2}, \Sigma^t Y) = \pi_0 \mathbf{Spec}^{\mathrm{op}}(A_{s+2}, \Sigma^{t+1} Y)$ is a (co)cycle and does not depend on the choices, up to (co)boundaries, and thus defines an element:*

$$d_{(2)}(x) \in \mathrm{Ext}_{\mathfrak{A}}^{s+2,t+1}(H^*X, H^*Y).$$

Moreover, this function

$$d_{(2)} \colon \mathrm{Ext}_{\mathfrak{A}}^{s,t}(H^*X, H^*Y) \to \mathrm{Ext}_{\mathfrak{A}}^{s+2,t+1}(H^*X, H^*Y)$$

is the Adams differential d_2.

Proof. This is [3, Theorems 4.2 and 7.3], or the case $n = 1, m = 3$ of [6, Theorem 15.11].

Here we used the natural isomorphism:

$$\mathrm{Ext}_{\pi_0 \mathbf{EM}^{\mathrm{op}}}^{i,j}(H^*X, H^*Y) \cong \mathrm{Ext}_{\mathfrak{A}}^{i,j}(H^*X, H^*Y)$$

where the left-hand side is defined as in Example 8.4. Using the equivalence of categories $H^* \colon \pi_0 \mathbf{EM}^{\mathrm{op}} \xrightarrow{\cong} \mathbf{Mod}_{\mathfrak{A}}^{\mathrm{fin}}$, this natural isomorphism follows from the natural isomorphisms:

$$\begin{aligned} \pi_0 \mathbf{Spec}^{\mathrm{op}}(A_{s+2}, \Sigma^{t+1} Y) &= \mathrm{Hom}_{\mathfrak{A}} \left(F_{s+2}, H^* \Sigma^{t+1} Y \right) \\ &= \mathrm{Hom}_{\mathfrak{A}} \left(F_{s+2}, \Sigma^{t+1} H^* Y \right). \end{aligned}$$

Cocycles modulo coboundaries in this group are precisely $\mathrm{Ext}_{\mathfrak{A}}^{s+2,t+1}(H^*X, H^*Y)$. □

Now assume that $d_2(x) = 0$ holds, so that x survives to the E_3-term. Since the obstruction

$$\mathcal{O}(\gamma, \gamma_s) = (\gamma \otimes d_{s+1}) \,\square\, (x'' \otimes \gamma_s)^{\boxminus}$$

vanishes, one can choose a left 2-track $\xi \colon \gamma \otimes d_{s+1} \Rightarrow x'' \otimes \gamma_s$, which makes (7.1) into a tertiary pre-chain complex in $\Pi_{(1,2)} \mathbf{Spec}^{\mathrm{op}}$. Since (A, d, γ, ξ) was a tertiary chain complex to begin with, the obstructions $\mathcal{O}(\xi_i, \xi_{i+1})$ are trivial, for $i \geq s$.

Theorem 7.3. *The obstruction $\mathcal{O}(\xi, \xi_s) \in \pi_2 \mathbf{Spec}^{\mathrm{op}}(A_{s+3}, \Sigma^t Y) = \pi_0 \mathbf{Spec}^{\mathrm{op}}(A_{s+3}, \Sigma^{t+2} Y)$ is a (co)cycle and does not depend on the choices up to (co)boundaries, and thus defines an element:*

$$d_{(3)}(x) \in E_3^{s+3,t+2}(X, Y).$$

Moreover, this function

$$d_{(3)} \colon E_3^{s,t}(X, Y) \to E_3^{s+3,t+2}(X, Y)$$

is the Adams differential d_3.

Proof. This is the case $n = 2, m = 4$ of [6, Theorem 15.11]. More precisely, by Theorem 9.3, the tertiary chain complex (A, d, γ, ξ) in $\Pi_{(1,2)} \mathbf{Spec}^{\mathrm{op}}$ yields a 2^{nd} order chain complex in $\mathrm{Nul}_2 \mathbf{Spec}^{\mathrm{op}}$ based on the same $\mathbf{EM}$-resolution $A_\bullet \to X$ in $\pi_0 \mathbf{Spec}^{\mathrm{op}}$. The construction of $d_{(3)}$ above corresponds to the construction d_3 in [6, Definition 15.8]. □

Remark 7.4. The groups $E_3^{s,t}(X, Y)$ are an instance of the secondary Ext groups defined in [3, §4]. Likewise, the next term $E_4^{s,t}(X, Y) = \ker d_{(3)} / \operatorname{im} d_{(3)}$ is a higher order Ext group as defined in [6, §15].

Theorem 7.5. *A weak equivalence of* 2*-track algebras induces an isomorphism of higher* Ext *groups, compatible with the differential* $d_{(3)}$*. More precisely, let* $F\colon \mathcal{A} \to \mathcal{A}'$ *be a weak equivalence between* 2*-track algebras* $\mathcal{A}$ *and* $\mathcal{A}'$ *which are weakly equivalent to* $\Pi_{(1,2)}\mathbf{EM}\{X,Y\}^{\mathrm{op}}$*. Then* F *induces isomorphisms* $E_{3,\mathcal{A}}^{s,t}(X,Y) \xrightarrow{\cong} E_{3,\mathcal{A}'}^{s,t}(FX,FY)$ *making the diagram*

$$\begin{array}{ccc} E_{3,\mathcal{A}}^{s,t}(X,Y) & \xrightarrow{d_{(3),\mathcal{A}}} & E_{3,\mathcal{A}}^{s+3,t+2}(X,Y) \\ \cong\downarrow & & \cong\downarrow \\ E_{3,\mathcal{A}'}^{s,t}(FX,FY) & \xrightarrow{d_{(3),\mathcal{A}'}} & E_{3,\mathcal{A}'}^{s+3,t+2}(FX,FY) \end{array}$$

commute. Here the additional subscript $\mathcal{A}$ *or* $\mathcal{A}'$ *denotes the ambient* 2*-track category in which the secondary* Ext *groups and the differential are defined.*

Proof. This follows from the case $n = 2$ of [6, Theorem 15.9], or an adaptation of the proof of [3, Theorem 5.1]. □

8. Higher order resolutions

In this section, we specialize some results of [6] about higher order resolutions to the case $n = 2$. We use the fact that a 2-track algebra has an underlying algebra of left 2-cubical balls, which is the topic of Section 9.

First, we recall some background on *relative* homological algebra; more details can be found in [3, §1].

Definition 8.1. Let $\mathbf{A}$ be an additive category and $\mathbf{a} \subseteq \mathbf{A}$ a full additive subcategory.

(1) A chain complex (A,d) is **a-exact** if for every object X of $\mathbf{a}$ the chain complex $\mathrm{Hom}_{\mathbf{A}}(X, A_\bullet)$ is an exact sequence of abelian groups.
(2) A chain map $f\colon (A,d) \to (A',d')$ is an **a-equivalence** if for every object X of $\mathbf{a}$, the chain map $\mathrm{Hom}_{\mathbf{A}}(X,f)$ is a quasi-isomorphism.
(3) For an object A of $\mathbf{A}$, an A**-augmented chain complex** $A_\bullet^\epsilon$ is a chain complex of the form

$$\cdots \longrightarrow A_1 \xrightarrow{d_0} A_0 \xrightarrow{\epsilon} A \longrightarrow 0 \longrightarrow \cdots$$

i.e., with $A_{-1} = A$ and $A_n = 0$ for $n < -1$. Such a complex can be viewed as a chain map $\epsilon\colon A_\bullet \to A$ where A is a chain complex concentrated in degree 0. The map $\epsilon = d_{-1}$ is called the **augmentation**.
(4) An **a-resolution** of A is an A-augmented chain complex $A_\bullet^\epsilon$ which is $\mathbf{a}$-exact and such that for all $n \geq 0$, the object A_n belongs to $\mathbf{a}$. In other words, an $\mathbf{a}$-resolution of A is a chain complex $A_\bullet$ in $\mathbf{a}$ together with an $\mathbf{a}$-equivalence $\epsilon\colon A_\bullet \to A$.

Example 8.2. Consider the category $\mathbf{A} = \mathbf{Mod}_R$ of R-modules for some ring R, and the subcategory $\mathbf{a}$ of free (or projective) R-modules. This recovers the usual homological algebra of R-modules.

Definition 8.3. Let $\mathcal{A}$ be an abelian category and $F\colon \mathbf{A} \to \mathcal{A}$ an additive functor. The **a-relative left derived functors** of F are the functors $L_n^{\mathbf{a}}F\colon \mathbf{A} \to \mathcal{A}$ for $n \geq 0$ defined by

$$(L_n^{\mathbf{a}}F)A = H_n\left(F(A_\bullet)\right)$$

where $A_\bullet \to A$ is any $\mathbf{a}$-resolution of A.

Likewise, if $F\colon \mathbf{A}^{\mathrm{op}} \to \mathcal{A}$ is a contravariant additive functor, its **a-relative right derived functors** of F are defined by

$$(R^n_{\mathbf{a}} F)A = H^n\left(F(A_\bullet)\right).$$

Example 8.4. The **a**-relative Ext groups are given by

$$\mathrm{Ext}^n_{\mathbf{a}}(A,B) := \left(R^n_{\mathbf{a}} \operatorname{Hom}_{\mathbf{A}}(-,B)\right)(A) = H^n \operatorname{Hom}_{\mathbf{A}}(A_\bullet, B).$$

Proposition 8.5 (Correction of 1-tracks)**.** *Let $\mathbf{B}$ be a category enriched in pointed groupoids, such that its homotopy category $\pi_0\mathbf{B}$ is additive. Let $\mathbf{a} \subseteq \pi_0\mathbf{B}$ be a full additive subcategory. Let (A,d,γ) be a secondary pre-chain complex in $\mathbf{B}$ based on an $\mathbf{a}$-resolution $A_\bullet \to X$ of an object X in $\pi_0\mathbf{B}$. Then there exists a secondary chain complex (A,d,γ') in $\mathbf{B}$ with the same objects A_i and differentials d_i. In particular (A,d,γ') is also based on the $\mathbf{a}$-resolution $A_\bullet \to X$.*

Proof. This follows from an adaptation of the proof of [3, Lemma 2.14], or the case $n = 1$ of [6, Theorem 13.2]. □

Proposition 8.6 (Correction of 2-tracks)**.** *Let $\mathcal{A}$ be a 2-track algebra such that its homotopy category $\pi_0\mathcal{A}$ is additive. Let $\mathbf{a} \subseteq \pi_0\mathcal{A}$ be a full additive subcategory. Let (A,d,γ,ξ) be a tertiary pre-chain complex in $\mathcal{A}$ based on an $\mathbf{a}$-resolution $A_\bullet \to X$ of an object X in $\pi_0\mathcal{A}$. Then there exists a tertiary chain complex (A,d,γ,ξ') in $\mathcal{A}$ with the same objects A_i, differentials d_i, and left paths γ_i. In particular, (A,d,γ,ξ') is also based on the $\mathbf{a}$-resolution $A_\bullet \to X$.*

Proof. This follows from the case $n = 2$ of [6, Theorem 13.2]. □

Theorem 8.7 (Resolution Theorem)**.** *Let $\mathcal{A}$ be a 2-track algebra such that its homotopy category $\pi_0\mathcal{A}$ is additive. Let $\mathbf{a} \subseteq \pi_0\mathcal{A}$ be a full additive subcategory. Let $A_\bullet \to X$ be an $\mathbf{a}$-resolution in $\pi_0\mathcal{A}$. Then there exists a tertiary chain complex in $\mathcal{A}$ based on the resolution $A_\bullet \to X$.*

Proof. This follows from the resolution theorems [6, Theorems 8.2 and 14.5]. □

9. Algebras of left 2-cubical balls

Proposition 9.1. *Every left cubical ball of dimension 2 is equivalent to C_k for some $k \geq 2$, where $C_k = B_1 \cup \cdots \cup B_k$ is the left cubical ball of dimension 2 consisting of k closed 2-cells going cyclically around the vertex 0, with one common 1-cell e_i between successive 2-cells B_i and B_{i+1}, where by convention $B_{k+1} := B_1$.*

See Figure 1, which is taken from [6, Figure 3].

Proof. Let B be a left cubical ball of dimension 2. For each closed 2-cell B_i, equipped with its homeomorphism $h_i\colon I^2 \xrightarrow{\cong} B_i$, the faces $\partial_1^1 B_i$ and $\partial_2^1 B_i$ are required to be 1-cells of the boundary $\partial B \cong S^1$, while the faces $\partial_1^0 B_i$ and $\partial_2^0 B_i$ are not in ∂B, and therefore must be faces of some other 2-cells. In other words, we have $\partial_1^0 B_i = \partial_1^0 B_j$ or $\partial_1^0 B_i = \partial_2^0 B_j$ for some other 2-cell B_j, in fact a unique B_j, because B is homeomorphic to a 2-disk.

Pick any 2-cell of B and call it B_1. Then the face $e_1 := \partial_2^0 B_1$ appears as a face of exactly one other 2-cell, which we call B_2. The remaining face e_2 of B_2 appears as a face of exactly one other 2-cell, which we call B_3. Repeating this process, we list distinct 2-cells $B_1, \ldots, B_k$, and B_{k+1} is one of the previously labeled 2-cells. Then B_{k+1} must be B_1, with $e_k = \partial_1^0 B_1$, since a 1-cell cannot appear as a common face of three 2-cells. Finally, this process exhausts all 2-cells, because all 2-cells share the common vertex 0, which has a neighborhood homeomorphic to an open 2-disk. □

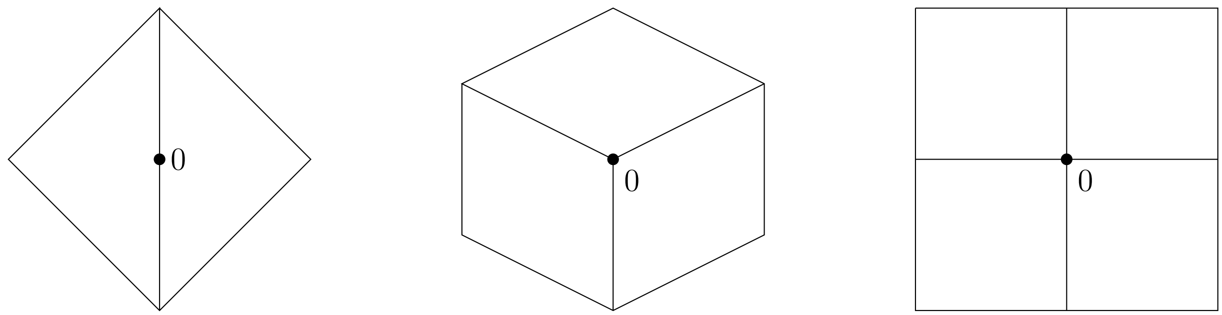


FIGURE 1. The left cubical balls C_2, C_3, and C_4.

Proposition 9.2. *A left* 2*-cubical ball (*[6, Definition 10.1]*) in a pointed space X corresponds to a circular chain of composable left* 2*-tracks:*

$$a = a_0 \xrightarrow{\alpha_1^{\epsilon_1}} a_1 \xrightarrow{\alpha_2^{\epsilon_2}} \cdots \to a_{k-1} \xrightarrow{\alpha_k^{\epsilon_k}} a_k = a$$

where the sign $\epsilon_i = \pm 1$ is the orientation of the 2*-cells in the left cubical ball (*[6, Definition 10.8]*). Moreover, such an expression $(\alpha_1, \dots, \alpha_k)$ of a left* 2*-cubical ball is unique up to cyclic permutation of the k left* 2*-tracks α_i. For example, $(\alpha_1, \alpha_2, \dots, \alpha_k)$ and $(\alpha_2, \dots, \alpha_k, \alpha_1)$ represent the same left* 2*-cubical ball. See Figure 2.*

Proof. By our convention for the $\square$-composition, a left 2-track α defines a morphism between left paths $\alpha\colon d_1^0\alpha \Rightarrow d_2^0\alpha$. The gluing condition for a left 2-cubical ball $(\alpha_1, \dots, \alpha_k)$ based on a left cubical ball $B = B_1 \cup \cdots \cup B_k$ as in Proposition 9.1 is that the restrictions $\alpha_i|_{e_i}$ and $\alpha_{i+1}|_{e_i}$ agree on the common edge $e_i \subset B_i \cap B_{i+1}$. This is the composability condition for $\alpha_{i+1}^{\epsilon_{i+1}} \square \alpha_i^{\epsilon_i}$. Indeed, up to a global sign, the sign of B_i is

$$\epsilon_i = \begin{cases} +1 & \text{if } e_i = \partial_2^0 B_i \\ -1 & \text{if } e_i = \partial_1^0 B_i \end{cases}$$

so that we have $\alpha_i^{\epsilon_i}\colon \alpha_i|_{e_{i-1}} \Rightarrow \alpha_i|_{e_i}$ and we may take $a_i = \alpha_i|_{e_i}$. □

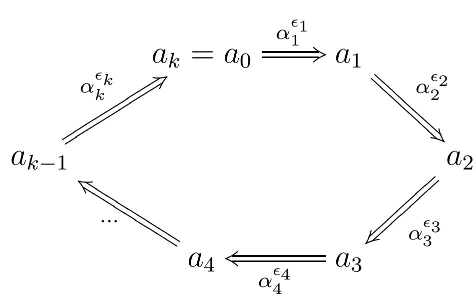


FIGURE 2. A left 2-cubical ball.

Theorem 9.3. (1) *A 2-track algebra $\mathcal{A}$ yields an algebra of left 2-cubical balls (*[6, Definition 11.1]*) in the following way. Consider the system $\Theta(\mathcal{A}) := \left((\mathcal{A}_{(1,2)}, \otimes), \pi_0\mathcal{A}, D, \mathcal{O}\right)$, where:*

- *$(\mathcal{A}_{(1,2)}, \otimes)$ is the underlying 2-graded category of $\mathcal{T}$ (described in Definition 5.1).*
- *$\pi_0\mathcal{A}$ is the homotopy category of $\mathcal{A}$.*
- *$q\colon (\mathcal{A})^0 = \mathcal{A}_{(1)0} \twoheadrightarrow \pi_0\mathcal{A}$ is the canonical quotient functor.*
- *$D\colon (\pi_0\mathcal{A})^{\mathrm{op}} \times \pi_0\mathcal{A} \to \mathbf{Ab}$ is the functor defined by $D(A,B) = \pi_2\mathcal{A}_{(1,2)}(A,B)$.*
- *The obstruction operator $\mathcal{O}$ is obtained by concatenating the corresponding left 2-tracks and using the structural isomorphisms ψ of the mapping 2-track groupoid:*

$$\mathcal{O}_B(\alpha_1, \alpha_2, \ldots, \alpha_k) = \psi_a \left(\alpha_k^{\epsilon_k} \square \cdots \square\, \alpha_2^{\epsilon_2} \square\, \alpha_1^{\epsilon_1}\right) \in \mathrm{Aut}_{\mathcal{A}_{(2)}(A,B)}(0) = \pi_2\mathcal{A}_{(1,2)}(A,B)$$

where we denoted $a = \delta_0\alpha_1 = \delta_1\alpha_k$.

(2) *Given a category $\mathcal{C}$ enriched in pointed spaces, $\Theta\left(\Pi_{(1,2)}\mathcal{C}\right)$ is the algebra of left 2-cubical balls*

$$(\mathrm{Nul}_2\, \mathcal{C}, \pi_0\mathcal{C}, \pi_2\mathcal{C}(-,-), \mathcal{O})$$

described in [6, §11]*.*

(3) *The construction Θ sends a tertiary pre-chain complex (A, d, δ, ξ) in $\mathcal{A}$ to a 2^{nd} order pre-chain complex in $\Theta(\mathcal{A})$, in the sense of* [6, Definition 11.4]*. Moreover, (A, d, δ, ξ) is a tertiary chain complex if and only if the corresponding 2^{nd} order pre-chain complex in $\Theta(\mathcal{A})$ is a 2^{nd} order chain complex.*

Proof. Let us check that the obstruction operator $\mathcal{O}$ is well-defined. By 9.2, the only ambiguity is the starting left 1-cube a_i in the composition. Two such compositions are conjugate in the groupoid $\mathcal{A}_{(2)}(A,B)$:

$$\begin{aligned} &\alpha_{i-1}^{\epsilon_{i-1}} \square \cdots \square\, \alpha_2^{\epsilon_2} \square\, \alpha_1^{\epsilon_1} \square\, \alpha_k^{\epsilon_k} \square \cdots \square\, \alpha_{i+1}^{\epsilon_{i+1}} \square\, \alpha_i^{\epsilon_i} \\ =& \left(\alpha_{i-1}^{\epsilon_{i-1}} \square \cdots \square\, \alpha_1^{\epsilon_1}\right) \square\, \alpha_k^{\epsilon_k} \square \cdots \square\, \alpha_{i+1}^{\epsilon_{i+1}} \square\, \alpha_i^{\epsilon_i} \square \cdots \square\, \alpha_1^{\epsilon_1} \square \left(\alpha_{i-1}^{\epsilon_{i-1}} \square \cdots \square\, \alpha_1^{\epsilon_1}\right)^{\boxminus} \\ =& \beta^{\boxminus} \square\, \alpha_k^{\epsilon_k} \square \cdots \square\, \alpha_1^{\epsilon_1} \square\, \beta \end{aligned}$$

with $\beta = \left(\alpha_{i-1}^{\epsilon_{i-1}} \square \cdots \square\, \alpha_1^{\epsilon_1}\right)^{\boxminus} : a_i \Rightarrow a_0$. Since $\mathcal{A}_{(2)}(A,B)$ is a strictly abelian groupoid, we have the commutative diagram:

$$\begin{array}{ccc} \mathrm{Aut}(a_0) & \xrightarrow{\varphi^\beta} & \mathrm{Aut}(a_i) \\ & \searrow^{\psi_{a_0}} & \downarrow{\scriptstyle \psi_{a_i}} \\ & & \mathrm{Aut}(0) \end{array}$$

so that $\mathcal{O}_B(\alpha_1, \ldots, \alpha_k)$ is well-defined.

The remaining properties listed in [6, Definition 11.1] are straightforward verifications. □

Appendix A. Models for homotopy 2-types

Recall that the left n-cubical set $\mathrm{Nul}_n(X)$ of a pointed space X depends only on the n-type P_nX of X [6, §1]. In particular the fundamental 2-track groupoid $\Pi_{(1,2)}(X)$ depends only on the 2-type P_2X of X. There are various algebraic models for homotopy 2-types in the literature, using 2-dimensional categorical structures. Let us mention the weak 2-groupoids of [15], the bigroupoids of [12], the double groupoids of [10], the two-typical double groupoids of [7], and the double groupoids with filling condition of [11].

In contrast, 2-track groupoids are *not* models for homotopy 2-types, not even of connected homotopy 2-types. In the application we are pursuing, the functor $\Pi_{(1,2)}$ will be applied to topological abelian groups, hence products of Eilenberg-MacLane spaces. We are *not* trying to encode the homotopy 2-type of the Eilenberg-MacLane mapping theory, but rather as little information as needed in order to compute the Adams differential d_3.

The fundamental 2-track groupoid $\Pi_{(1,2)}(X)$ encodes the 1-type of X, via the fundamental groupoid $\Pi_{(1)}(X)$. Moreover, as noted in Remark 3.13, it also encodes the homotopy group $\pi_2(X)$. However, it fails to encode the $\pi_1(X)$-action on $\pi_2(X)$, as we will show below.

A.1. **Connected 2-track groupoids.** Recall that a category $\mathcal{C}$ is called *skeletal* if any isomorphic objects are equal. A *skeleton* of $\mathcal{C}$ is a full subcategory on a collection consisting of one representative object in each isomorphism class of objects of $\mathcal{C}$. Every groupoid is equivalent to a disjoint union of groups, that is, a coproduct of single-object groupoids. The inclusion $\mathrm{sk}G \xrightarrow{\cong} G$ of a skeleton of G provides such an equivalence. A similar construction yields the following statement for 2-track groupoids.

Lemma A.1. *Let $G = \left(G_{(1)}, G_{(2)}\right)$ be a 2-track groupoid.*

1. *There is a weak equivalence of 2-track groupoids $\mathrm{sk}_{(1)}G \xrightarrow{\sim} G$ where the first groupoid of $\mathrm{sk}_{(1)}G$ is skeletal.*
2. *If G is connected and $G_{(1)}$ is skeletal, then there is a weak equivalence of 2-track groupoids $\mathrm{sk}_{(2)}G \xrightarrow{\sim} G$ where both groupoids of $\mathrm{sk}_{(2)}G$ are skeletal.*

In particular, if G is connected, then $\mathrm{sk}_{(2)}\mathrm{sk}_{(1)}G \xrightarrow{\sim} \mathrm{sk}_{(1)}G \xrightarrow{\sim} G$ is a weak equivalence between G and a 2-track groupoid whose constituent groupoids are both skeletal.

Lemma A.2. *Let G and G' be connected 2-track groupoids whose constituent groupoids are skeletal. If there are isomorphisms of homotopy groups $\varphi_1 \colon \pi_1 G \simeq \pi_1 G'$ and $\varphi_2 \colon \pi_2 G \simeq \pi_2 G'$, then there is a weak equivalence $\varphi \colon G \xrightarrow{\sim} G'$.*

Proof. Since $G_{(1)}$ and $G'_{(1)}$ are skeletal, they are in fact groups, and the group isomorphism φ_1 is an isomorphism of groupoids $\varphi_{(1)} \colon G_{(1)} \xrightarrow{\cong} G'_{(1)}$.

Now we define a functor $\varphi_{(2)} \colon G_{(2)} \to G'_{(2)}$. On objects, it is given by the composite

$$G_{(2)0} = \mathrm{Comp}\, G_{(2)} \xrightarrow[\simeq]{q} \mathrm{Star}\, G_{(1)} = G_{(1)}(0,0) = \pi_1 G \xrightarrow{\qquad\qquad}$$

$$\xrightarrow[\simeq]{\varphi_1} \pi_1 G' = G'_{(1)}(0,0) = \mathrm{Star}\, G'_{(1)} \xleftarrow[\simeq]{q} \mathrm{Comp}\, G'_{(2)} = G'_{(2)0}$$

which is a bijection. On morphisms, $\varphi_{(2)}$ is defined as follows. We have $G_{(2)}(a,b) = \emptyset$ when $a \neq b$, so there is nothing to define then. On the automorphisms of an object $a \in G_{(2)0}$, define $\varphi_{(2)}$ as the composite

$$G_{(2)}(a,a) = \mathrm{Aut}_{G_{(2)}}(a) \xrightarrow[\simeq]{\psi_a} \mathrm{Aut}_{G_{(2)}}(0) = \pi_2 G \xrightarrow{\qquad\qquad\qquad}$$

$$\xrightarrow[\simeq]{\varphi_2} \pi_2 G' = \mathrm{Aut}_{G'_{(2)}}(0') \xleftarrow[\simeq]{\psi'_{\varphi(a)}} \mathrm{Aut}_{G'_{(2)}}(\varphi(a)) = G'_{(2)}(\varphi(a), \varphi(a)).$$

Then $\varphi_{(2)}$ is a functor and commutes with the structural isomorphisms, by construction. Thus $\varphi = (\varphi_{(1)}, \varphi_{(2)}) \colon G \to G'$ is a morphism of 2-track groupoids, and is moreover a weak equivalence. $\square$

Corollary A.3. *Let G and G' be connected 2-track groupoids with isomorphic homotopy groups $\pi_i G \simeq \pi_i G'$ for $i = 1, 2$. Then G and G' are weakly equivalent, i.e., there is a zigzag of weak equivalences between them.*

Proof. Consider the zigzag of weak equivalences

$$\begin{array}{ccc} G & & G' \\ \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\ \mathrm{sk}_{(1)} G & & \mathrm{sk}_{(1)} G' \\ \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\ \mathrm{sk}_{(2)}\mathrm{sk}_{(1)} G & \xrightarrow[\sim]{\varphi} & \mathrm{sk}_{(2)}\mathrm{sk}_{(1)} G' \end{array}$$

where the bottom morphism φ is obtained from Lemma A.2. $\square$

By Remark 3.13, the functor $\Pi_{(1,2)} \colon \mathbf{Top}_* \to \mathbf{Gpd_{(1,2)}}$ induces a functor

$$\Pi_{(1,2)} \colon \mathrm{Ho}\,(\mathbf{connected}\, 2{-}\mathbf{Types}) \to \mathrm{Ho}\left(\mathbf{Gpd_{(1,2)}}\right) \tag{A.1}$$

where the left-hand side denotes the homotopy category of connected 2-types (localized with respect to weak homotopy equivalences), and the right-hand side denotes the localization with respect to weak equivalences, as in Definition 3.12.

Proposition A.4. *The functor $\Pi_{(1,2)}$ in (A.1) is not an equivalence of categories.*

Proof. Let X and Y be connected 2-types with isomorphic homotopy groups π_1 and π_2, but distinct π_1-actions on π_2. Then X and Y are not weakly equivalent, but $\Pi_{(1,2)}(X)$ and $\Pi_{(1,2)}(Y)$ are weakly equivalent, by Corollary A.3. $\square$

A.2. **Comparison to bigroupoids.** Any algebraic model for (pointed) homotopy 2-types has an underlying 2-track groupoid. Using the globular description in Remark 3.1, the most direct comparison is to the bigroupoids of [12]. A *pointed* bigroupoid (resp. double groupoid) will mean one equipped with a chosen object, here denoted x_0 to emphasize that it is unrelated to the algebraic structure of the bigroupoid.

Proposition A.5. *Let $\Pi_2^{\mathrm{BiGpd}}(X)$ denote the homotopy bigroupoid of a space X constructed in* [12]*, where it was denoted $\Pi_2(X)$.*

1. *There is a forgetful functor U from pointed bigroupoids to 2-track groupoids.*
2. *For a pointed space X, there is a natural isomorphism of 2-track groupoids $\Pi_{(1,2)}(X) \cong U\Pi_2^{\mathrm{BiGpd}}(X)$.*

Proof. Let B be a bigroupoid. We construct a 2-track groupoid UB as follows. The first constituent groupoid of UB is the underlying groupoid of B

$$UB_{(1)} := \pi_0 B$$

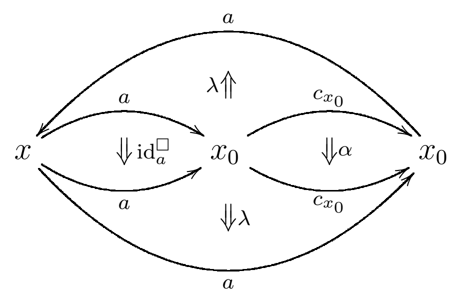
obtained by taking the components of each mapping groupoid $B(x,y)$. The second constituent groupoid of UB is a coproduct of mapping groupoids

$$UB_{(2)} := \coprod_{x \in \mathrm{Ob}(B)} B(x, x_0).$$

The quotient function $q\colon UB_{(2)0} \to \mathrm{Star}\, UB_{(1)}$ is induced by the natural quotient maps $\mathrm{Ob}\left(B(x,x_0)\right) \twoheadrightarrow \pi_0 B(x,x_0)$. To define the structural isomorphisms

$$\psi_a\colon \mathrm{Aut}(a) \xrightarrow{\cong} \mathrm{Aut}(c_{x_0})$$

for objects $a \in UB_{(2)0}$, which are 1-morphisms to the basepoint $a\colon x \to x_0$, consider the diagram



where $\lambda\colon c_{x_0} \bullet a \Rightarrow a$ is the *left identity* coherence 2-isomorphism, $\bullet$ denotes composition of 1-morphisms, and c_{x_0} is the identity 1-morphism of the object x_0. (We kept our notation $\square$ for composition of 2-morphisms.) The inverse $\psi_a^{-1}\colon \mathrm{Aut}(c_{x_0}) \to \mathrm{Aut}(a)$ is defined by going from top to bottom in the diagram, namely

$$\psi_a^{-1}(\alpha) = \lambda \square \left(\alpha \bullet \mathrm{id}^{\square}_a\right) \square \lambda^{\boxminus}.$$

One readily checks that UB is a 2-track groupoid, that this construction U is functorial, and that $U\Pi_2^{\mathrm{BiGpd}}(X)$ is naturally isomorphic to $\Pi_{(1,2)}(X)$ as 2-track groupoids. $\square$

A.3. **Comparison to double groupoids.** The homotopy double groupoid $\rho_2^{\square}(X)$ from [10] is a cubical construction. Following the terminology therein, *double groupoid* will be shorthand for *edge symmetric double groupoid with connection.*

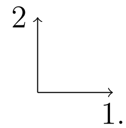
Let us recall the geometric idea behind $\rho_2^{\square}(X)$. A path $a\colon I \to X$ has an underlying *semitrack* $\langle a \rangle$, defined as its equivalence class with respect to *thin* homotopy rel ∂I. A semitrack $\langle a \rangle$ in turn has an underlying track $\{a\}$. A square $u\colon I^2 \to X$ has an underlying 2-track $\{u\}$. A 2-track $\{u\}$ in turn has an underlying equivalence class $\{u\}_T$ with respect to *cubically thin homotopy*, i.e., a homotopy whose restriction to the boundary ∂I^2 is thin (not necessarily stationary). The homotopy double groupoid $\rho_2^{\square}(X)$ encodes semitracks $\langle a \rangle$ in X and 2-tracks $\{u\}_T$ up to cubically thin homotopy.

Proposition A.6. *Let $\rho_2^{\square}(X)$ denote the homotopy double groupoid of a space X constructed in* [10].

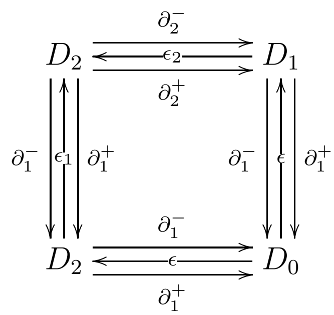
(1) *There is a forgetful functor U from pointed double groupoids to 2-track groupoids.*

(2) *For a pointed space X, there is a natural weak equivalence of 2-track groupoids $\Pi_{(1,2)}(X) \xrightarrow{\sim} U\rho_2^{\square}(X)$.*

Proof. We adopt the notation of [10], including that compositions in a double groupoid are written in diagrammatic order, i.e., $a+b$ denotes the composition $x \xrightarrow{a} y \xrightarrow{b} z$. However, we keep our graphical convention for the two axes:



Let D be a double groupoid, whose data is represented in the diagram of sets

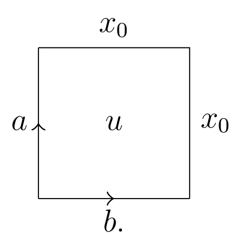


along with connections $\Gamma^-, \Gamma^+ \colon D_1 \to D_2$. Two 1-morphisms $a, b \in D_1$ with same endpoints $\partial_1^-(a) = \partial_1^-(b) = x$, $\partial_1^+(a) = \partial_1^+(b) = y$ are called *homotopic* if there exists a 2-morphism $u \in D_2$ satisfying $\partial_2^-(u) = a$, $\partial_2^+(u) = b$, $\partial_1^-(u) = \epsilon(x)$, $\partial_2^+(u) = \epsilon(y)$. We write $a \sim b$ if a and b are homotopic.

We now define the underlying 2-track groupoid UD. The first constituent groupoid $UD_{(1)}$ has object set D_0 and morphism set $D_1/\sim$, with groupoid structure inherited from the groupoid (D_0, D_1). The second constituent groupoid $UD_{(2)}$ has object set

$$UD_{(2)0} := \left\{ a \in D_1 \mid \partial_1^+(a) = x_0 \right\}.$$

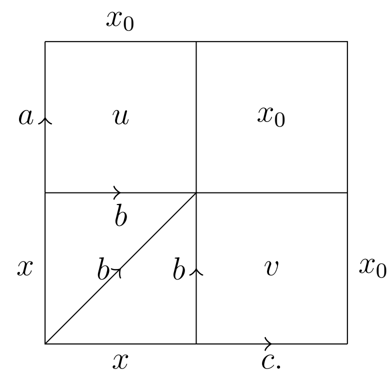
A morphism in $UD_{(2)}$ from a to b is an element $u \in D_2$ satisfying $\partial_1^-(u) = a$, $\partial_2^-(u) = b$, $\partial_1^+(u) = \epsilon(x_0)$, $\partial_2^+(u) = \epsilon(x_0)$, as illustrated here:



Composition in $UD_{(2)}$ is defined as follows. Given 1-morphisms $a, b, c \colon x \to x_0$ in D_1 and morphisms $u \colon a \Rightarrow b$ and $v \colon b \Rightarrow c$ in $UD_{(2)}$, their composition $v \square u \colon a \Rightarrow c$ is defined by

$$\begin{aligned} v \square u &= \left(\Gamma^+(b) +_2 u\right) +_1 (v +_2 \odot_{x_0}) \\ &= \left(\Gamma^+(b) +_2 u\right) +_1 v \\ &= \left(\Gamma^+(b) +_1 v\right) +_2 u \end{aligned}$$

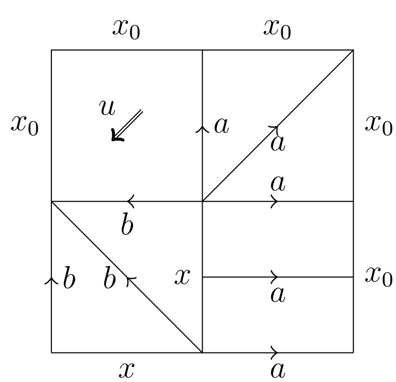
as illustrated here:



The identity morphisms in $UD_{(2)}$ are given by $\mathrm{id}_a^{\square} = \Gamma^-(a)$. The inverse of $u\colon a \Rightarrow b$ is given by

$$\begin{aligned} u^{\boxminus} &= \big((-_1)\Gamma^+(b) +_2 (-_1)u\big) +_1 \big(\epsilon_2(a) +_2 \Gamma^-(a)\big) \\ &= \big((-_1)\Gamma^+(b) +_2 (-_1)u\big) +_1 \Gamma^-(a) \end{aligned}$$

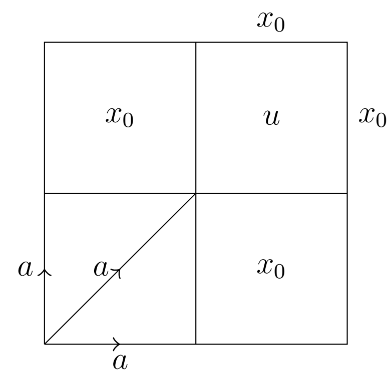
as illustrated here:



The structural isomorphisms $\psi_a^{-1}\colon \mathrm{Aut}(\epsilon(x_0)) \to \mathrm{Aut}(a)$ are defined by

$$\begin{aligned} \psi_a^{-1}(u) &= \big(\Gamma^-(a) +_2 \odot_{x_0}\big) +_1 (\odot_{x_0} +_2 u) \\ &= \Gamma^-(a) +_1 u \\ &= \Gamma^-(a) +_2 u \end{aligned}$$

as illustrated here:



The quotient function $q\colon UD_{(2)0} \twoheadrightarrow \operatorname{Star} UD_{(1)}$ is induced by the quotient function $D_1 \twoheadrightarrow D_1/\sim$. One readily checks that UD is a 2-track groupoid, and that this construction U is functorial.

For a pointed space X, define a comparison map $\Pi_{(1,2)}(X) \to U\rho_2^{\square}(X)$ which is an isomorphism on $\Pi_{(1)}(X)$, and which quotients out the thin homotopy relation between left paths in X and the cubically thin homotopy relation between left 2-tracks. This defines a natural weak equivalence of 2-track groupoids. □

References

[1] J. F. Adams, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. **32** (1958), 180–214. MR0096219 (20 #2711)

[2] H.-J. Baues, *The algebra of secondary cohomology operations*, Progress in Mathematics, vol. 247, Birkhäuser Verlag, Basel, 2006. MR2220189 (2008a:55015)

[3] H.-J. Baues and M. Jibladze, *Secondary derived functors and the Adams spectral sequence*, Topology **45** (2006), no. 2, 295 –324, DOI 10.1016/j.top.2005.08.001. MR2193337 (2006k:55031)

[4] H. J. Baues and D. Blanc, *Stems and spectral sequences*, Algebr. Geom. Topol. **10** (2010), no. 4, 2061–2078, DOI 10.2140/agt.2010.10.2061. MR2728484 (2012c:55017)

[5] H.-J. Baues and M. Jibladze, *Dualization of the Hopf algebra of secondary cohomology operations and the Adams spectral sequence*, J. K-Theory **7** (2011), no. 2, 203 –347, DOI 10.1017/is010010029jkt133. MR2787297 (2012h:55023)

[6] H.-J. Baues and D. Blanc, *Higher order derived functors and the Adams spectral sequence*, J. Pure Appl. Algebra **219** (2015), no. 2, 199–239, DOI 10.1016/j.jpaa.2014.04.018. MR3250522

[7] D. Blanc and S. Paoli, *Two-track categories*, J. K-Theory **8** (2011), no. 1, 59–106, DOI 10.1017/is010003020jkt116. MR2826280 (2012h:18021)

[8] A. K. Bousfield and D. M. Kan, *The homotopy spectral sequence of a space with coefficients in a ring*, Topology **11** (1972), 79–106. MR0283801 (44 #1031)

[9] A. K. Bousfield and E. M. Friedlander, *Homotopy theory of* Γ*-spaces, spectra, and bisimplicial sets*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, pp. 80–130. MR513569 (80e:55021)

[10] R. Brown, K. A. Hardie, K. H. Kamps, and T. Porter, *A homotopy double groupoid of a Hausdorff space*, Theory Appl. Categ. **10** (2002), 71–93. MR1883479 (2003d:18010)

[11] A. M. Cegarra, B. A. Heredia, and J. Remedios, *Double groupoids and homotopy 2-types*, Appl. Categ. Structures **20** (2012), no. 4, 323–378, DOI 10.1007/s10485-010-9240-1. MR2943635

[12] K. A. Hardie, K. H. Kamps, and R. W. Kieboom, *A homotopy bigroupoid of a topological space*, Appl. Categ. Structures **9** (2001), no. 3, 311–327, DOI 10.1023/A:1011270417127. MR1836257 (2002f:18011)

[13] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512, DOI 10.1112/S0024611501012692. MR1806878 (2001k:55025)

[14] C. R. F. Maunder, *On the differentials in the Adams spectral sequence*, Proc. Cambridge Philos. Soc. **60** (1964), 409–420. MR0167980 (29 #5245)

[15] Z. Tamsamani, *Sur des notions de n-catégorie et n-groupoïde non strictes via des ensembles multi-simpliciaux*, K-Theory **16** (1999), no. 1, 51–99, DOI 10.1023/A:1007747915317 (French, with English summary). MR1673923 (99m:18007)

E-mail address: `baues@mpim-bonn.mpg.de`

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

E-mail address: `franklan@mpim-bonn.mpg.de`

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany